

Distinguishing extension numbers for \mathbf{R}^n and S^n

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Abstract

In the setting of a group Γ acting faithfully on a set X , a k -coloring $c : X \rightarrow \{1, 2, \dots, k\}$ is called Γ -distinguishing if the only element of Γ that fixes c is the identity element. The distinguishing number $D_\Gamma(X)$ is the minimum value of k such that a Γ -distinguishing k -coloring of X exists. Now, fixing $k = D_\Gamma(X)$, a subset $W \subset X$ with trivial pointwise stabilizer satisfies the precoloring extension property $P(W)$ if every precoloring $c : X - W \rightarrow \{1, \dots, k\}$ can be extended to a Γ -distinguishing k -coloring of X . The distinguishing extension number $\text{ext}_D(X, \Gamma)$ is then defined to be the minimum n such that for all applicable $W \subset X$, $|W| \geq n$ implies that $P(W)$ holds. In this paper, we compute $\text{ext}_D(X, \Gamma)$ in two particular instances: when $X = S^1$ is the unit circle and $\Gamma = \text{Isom}(S^1) = O(2)$ is its isometry group, and when $X = V(C_n)$ is the set of vertices of the cycle of order n and $\Gamma = \text{Aut}(C_n) = D_n$, the dihedral group of a regular n -gon. This resolves two conjectures of Ferrara, Gethner, Hartke, Stolee, and Wenger. In the case of $X = \mathbf{R}^2$, we prove that $\text{ext}_D(\mathbf{R}^2, SE(2)) < \infty$, which is consistent with (but does not resolve) another conjecture of Ferrara et al. On the other hand, we also prove that for all $n \geq 3$, $\text{ext}_D(S^{n-1}, O(n)) = \infty$, and for all $n \geq 3$, $\text{ext}_D(\mathbf{R}^n, E(n)) = \infty$, disproving two other conjectures from the same authors.

1 Introduction

Let Γ be a group which acts faithfully on a set X . As defined by Tymoczko in [7], a k -coloring $c : X \rightarrow \{1, \dots, k\}$ is distinguishing with respect to Γ if the only $\gamma \in \Gamma$ for which $c \circ \gamma = c$ is the identity element (that is, no nontrivial action of some $\gamma \in \Gamma$ fixes the coloring). The distinguishing number of (X, Γ) , denoted $D_\Gamma(X)$, is defined to be the smallest k such that X has a Γ -distinguishing k -coloring. A special case of this introduced by Anderson and Collins in [1] takes $X = V(G)$ to be the vertices of a graph, and $\Gamma = \text{Aut}(G)$ to be the automorphism group of the graph. Of particular interest are the cases $G = C_n$, the cycle of order n . In [1], it is proved that $D(C_n) = 2$ for all $n \geq 6$, while $D(C_3) = D(C_4) = D(C_5) = 3$.

In [2], Ferrara, Gethner, Hartke, Stolee, and Wenger introduce a refinement to the distinguishing number problem, in the form of extending precolorings. For the rest of the paper, we fix $k = D_\Gamma(X)$. Then, given a subset $W \subset X$ and any precoloring $c : X - W \rightarrow \{1, \dots, k\}$, we can ask if it is possible to extend c to a Γ -distinguishing coloring $c^* : X \rightarrow \{1, \dots, k\}$. For convenience, we introduce the following notation.

Definiton. For $W \subset X$ such that the pointwise stabilizer $\text{stab}_\Gamma(W)$ is trivial, we define the *precoring extension property* $P(W)$ as follows: $P(W)$ holds if and only if every precoloring $c : X - W \rightarrow \{1, 2, \dots, k\}$ can be extended to a distinguishing k -coloring of X .

Based on this notion, in [2], the notion of a *distinguishing extension number* is introduced.

Definition. The *distinguishing extension number* $\text{ext}_D(X, \Gamma)$ is equal to the smallest value of n such that for all $W \subset X$, if $|W| \geq n$ and W is not pointwise stabilized by any nontrivial $\gamma \in \Gamma$, then $P(W)$ holds.

In this paper, we investigate $\text{ext}_D(\mathbf{R}^n, \text{Isom}(\mathbf{R}^n))$ and $\text{ext}_D(S^n, \text{Isom}(S^n))$, where S^n denotes the unit n -sphere; for the rest of the paper, we use $O(n)$ to denote $\text{Isom}(S^{n-1})$ and $E(n)$ to denote $\text{Isom}(\mathbf{R}^n)$.

In the first half of this paper, we compute $\text{ext}_D(X, \Gamma)$ in two particular cases. One case consists of the graph setting mentioned above; in particular, for $X = V(C_n)$ and $\Gamma = \text{Aut}(C_n) = D_n$, where C_n is the cycle of order n and D_n is the dihedral group of a regular n -gon. As mentioned earlier, we already know that for $n \geq 6$, C_n has distinguishing number equal to 2. The other case consists of $X = S^1$ and $\Gamma = O(2)$; it is easy to see that $D_{O(2)}(S^1) = 2$ as well. In [2], it was proved that $\text{ext}_D(\mathbf{R}, E(1)) = 4$, and some partial results on C_n and S^1 were given.

Theorem [2]. *If $n \geq 6$ is not divisible by 2, 3, or 5, then $\text{ext}_D(C_n) = 4$. Furthermore, $\text{ext}_D(S^1, O(2)) \leq 16$.*

The authors of [2] also conjectured the exact values for the extension numbers for S^1 and the remaining C_n , which I prove correct here.

Theorem 1. *Let $n \geq 6$. If 4 and 5 do not divide n , then $\text{ext}_D(C_n) = 4$. If $4 \mid n$ but $5 \nmid n$, then $\text{ext}_D(C_n) = 5$. If $5 \mid n$, then $\text{ext}_D(C_n) = 6$. Finally, $\text{ext}_D(S^1, O(2)) = 6$.*

The proof of this theorem involves considering general subsets $W \subset S^1$ of cardinality equal to 4 or 5 and investigating when $P(W)$ holds. In order to prove Theorem 1, we prove a somewhat stronger characterization of when $P(W)$ holds in this situation.

Proposition.

- i) Let $W \subset S^1$ and $|W| = 5$. Then, $P(W)$ holds unless W is the set of vertices of a regular pentagon.*
- ii) Let $W \subset S^1$, $|W| = 4$, and suppose that the four orbits of elements of W under the translation of order 5 are distinct. Then, $P(W)$ holds unless W is the set of vertices of a square.*

The proposition tells us that the only obstructions to extending all precolorings of $S^1 - \{\text{four points}\}$ are the obstructions due to symmetries of C_4 and C_5 .

In the second half of the paper, we consider what happens in higher dimensions; we present fairly concrete examples to demonstrate that for all $n \geq 3$, $\text{ext}_D(\mathbf{R}^n, E(n)) = \infty$ and $\text{ext}_D(S^n, O(n)) = \infty$. In fact, we prove a stronger result.

Theorem 2. *Let $n \geq 3$, $X = \mathbf{R}^n$ or S^n , and $\Gamma = \text{Isom}(X)$. Then, there exist uncountable sets $W \subset X$ which have trivial pointwise stabilizer inside Γ but do not satisfy $P(W, \Gamma)$.*

The extension number $\text{ext}_D(\mathbf{R}^3, E(3))$ was previously conjectured to be finite; in fact, Theorem 2 provides the first known instances in which $\text{ext}_D(X, \Gamma)$ is infinite. The question in two dimensions is harder to resolve; in the case of \mathbf{R}^2 , the authors of [2] conjectured the following.

Conjecture [2]. $\text{ext}_D(\mathbf{R}^2, E(2)) = 7$.

We obtain some partial results by considering subgroups of $E(2)$. Let $SE(2) : \{\vec{v} \mapsto A\vec{v} + \vec{b}, A \in SO(2), \vec{b} \in \mathbf{R}^2\}$. We prove the following theorem.

Theorem 3. $\text{ext}_D(\mathbf{R}^2, O(2)) = 7$, and $\text{ext}_D(\mathbf{R}^2, SE(2)) < \infty$.

In the case of S^2 , we are able to show that $\text{ext}_D(S^2, O(3)) = \infty$ using an entirely different argument from the arguments in higher dimensions. We prove the following theorem.

Theorem 4. *$P(W, SO(3))$ does not hold for any finite subset of S^2 . Furthermore, assuming the axiom of choice, $P(W, SO(3))$ does not hold for any countable subset of S^2 .*

Finally, after proving Theorem 4, we discuss a few unanswered questions regarding extending precolorings on \mathbf{R}^n and S^n .

2 Theorem 1: Extending precolorings on S^1

2.1 Preliminaries

Theorem 1 is concerned with computing $\text{ext}_D(S^1, O(2))$ and $\text{ext}_D(C_n, \text{Aut}(C_n))$. We note that the lower bound $\text{ext}_D(S^1) \geq 6$ was already proven in [2] (and the appropriate lower bounds for all of the C_n were also proven). This was done by embedding C_4 or C_5 (as appropriate) into S^1 and C_n , and observing that two colors are insufficient to distinguish C_4 and C_5 (so $P(C_4, S^1)$ and $P(C_5, S^1)$ do not hold). Therefore, we only need to show that the appropriate values also serve as upper bounds to the extension numbers.

First of all, we can quickly eliminate all dependence on C_n and work entirely over S^1 (see [2] for the complete framework). The vertices of C_n can be embedded into S^1 by a map ϕ which sends $\{1, 2, \dots, n\}$ to the n th roots of unity. Under this embedding, for any $W \subset C_n$, we have the following fact.

Fact. $P(\phi(W), S^1)$ implies $P(W, C_n)$.

This holds because the setwise stabilizer of $\phi(V(C_n))$ inside $O(2)$ is canonically isomorphic to $\text{Aut}(C_n)$. As a result, for the rest of this section, we will consider subsets $W \subset S^1$, and $P(W)$ will always be taken to be over S^1 . Furthermore, we have the following reduction.

Observation. For all $\gamma \in O(2)$, $P(\gamma W)$ holds if and only if $P(W)$ holds.

This is true because if the coloring $c : \gamma W \rightarrow \{R, B\}$ is preserved by $\gamma' \in O(2)$, then the coloring $c \circ \gamma : W \rightarrow \{R, B\}$ is preserved by $\gamma^{-1}\gamma'\gamma$.

This reduction will be used extensively throughout the rest of the paper in the following way: we say that two subsets W and W' of S^1 are $O(2)$ -equivalent (written $W \cong W'$) if $W' = \gamma W$ for some $\gamma \in O(2)$. This observation tells us that if $W \cong W'$, then we can always replace W with W' without loss of generality, in order to determine if $P(W)$ holds or not.

Finally, to establish notation for the rest of the proof, we identify $S^1 \cong \mathbf{R}/2\pi\mathbf{Z}$. We use σ to denote a translation, with σ_a denoting the map $x \mapsto x + a$, and use τ to denote a reflection, with τ_a denoting the map $x \mapsto -x + 2a$. If c is a 2-coloring, then we will use c_- to denote the opposite coloring to c (i.e., the unique coloring such that $c(x) \neq c_-(x)$ wherever c is defined).

2.2 An extension of [2] Theorem 7

In [2], the following theorem was proved.

Theorem [2, Theorem 7]. Suppose $W \subset S^1$ of cardinality 4 satisfies the following condition, denoted $T(W)$: the intersection $(W + \frac{i}{k}) \cap W = \emptyset$ for $2 \leq k \leq 5$, $1 \leq i \leq k - 1$. Then, $P(W)$ holds.

To prove this theorem, the authors prove as a lemma that $T(W)$ implies $R(W)$ where $R(W)$ is the following property: there exists $w_0 \in W$ such that $\tau_{w_0}(W - \{w_0\}) \cap W = \emptyset$. It is then proved that $T(W)$ and $R(W)$ together imply $P(W)$. The goal of Section 3.2 is to prove that $R(W)$ alone implies $P(W)$. Later, we will replace condition $T(W)$ with successively weaker translation conditions until we have proven Theorem 1.

The proof that $R(W)$ implies $P(W)$ is almost exactly the same as the proof of Theorem 7 in [2]; however, we need to substitute the following lemma for Lemma 4 in [2].

Lemma 2.2.1. *Suppose that $W \subset S^1$ of cardinality 4 satisfies $R(W)$, and we have a precoloring $c : S^1 - W \rightarrow \{R, B\}$. Then, there are at most six extensions of c to $S^1 - \{w_0\}$ which are preserved by either τ_{w_0} or a translation (called “forbidden” in [2]).*

Proof. Since $\tau_{w_0}(W - \{w_0\}) \cap W = \emptyset$, there is at most one extension of c to $S^1 - \{w_0\}$ which permits τ_{w_0} . Let $\sigma_{\frac{1}{2}}$ be the translation of order 2. Then, $\sigma_{\frac{1}{2}}(W) \neq W$, because if equality were to hold, property $R(W)$ would not be satisfied. Therefore, there are at most two extensions of c which are preserved by $\sigma_{\frac{1}{2}}$ (there may be two if $W = \{w_0, a, a + \frac{1}{2}, b\}$ where $b \neq \frac{1}{2} + w_0$). Furthermore, if c^* is an extension of c preserved by σ of even order, then it is also preserved by $\sigma_{\frac{1}{2}}$ [either clockwise or counterclockwise iteration of σ will avoid crossing w_0 , and shows us that $\sigma_{\frac{1}{2}}$ will in fact always preserve c^*].

Suppose c_1 and c_2 are extensions of c which permit σ_1 and σ_2 of odd or infinite order, and let $w \in W$ be such that $c_1(w) \neq c_2(w)$. We claim that at least one of σ_1 and σ_2 has order 3. On the contrary, suppose that neither σ_1 nor σ_2 had order 3. If $|\sigma_1| = |\sigma_2| < \infty$, then as in the Lemma 4 argument in [2], we let \mathcal{O}_w denote the σ_1 -orbit of w . In this situation, we may suppose that $\sigma_1 = \sigma_2$ (as σ_1 will always be some power of σ_2). Since $c_1(\mathcal{O}_w) \cap c_2(\mathcal{O}_w) = \emptyset$, we can conclude that $\mathcal{O}_2 \subset W$. But $|\sigma_1| > 4$ (because $|\sigma|$ is odd); since $|W| = 4$, we have a contradiction. Therefore, we may assume that $|\sigma_2| > |\sigma_1| > 3$, so $|\sigma_2| \geq 7$. From here, the argument from [2] applies (it is possible to find an element $x_{\mathcal{O}}$ of any σ_1 -orbit \mathcal{O} such that $x_{\mathcal{O}}$ and $\sigma_2(x_{\mathcal{O}})$ are not in W), and we obtain a contradiction. Thus, either σ_1 or σ_2 has order 3.

Now, suppose we have c_1, c_2 , and c_3 (of odd or infinite order) which permit σ_1, σ_2 , and σ_3 . By the previous paragraph, we obtain that without loss of generality, $|\sigma_1| = |\sigma_2| = 3$, which also means that without loss of generality, $\sigma_1 = \sigma_2$. Therefore, $|\sigma_1(W) \cap W| = 3$, and assuming $c_1 \neq c_2$, we conclude that $c_1 = c_2$ outside of $\sigma_1(W) \cap W$, while $c_1 = c_2$ on $\sigma_1(W) \cap W$. If we had a fourth extension c_4 , we would also have $|\sigma_4| = 3$, but then we would obtain that $c_4 = c_1$ or $c_4 = c_2$, a contradiction. Therefore, there are at most three extensions of c which permit a translation of order greater than two. In total, then, we have at most $3 + 2 + 1 = 6$ forbidden extensions, which proves Lemma 3.2. ■

Theorem 2.2.2. *If $W \subset S^1$ of cardinality 4 satisfies condition $R(W)$, then $P(W)$ holds.*

Proof. Suppose that $W \subset S^1$, $|W| = 4$, and $R(W)$ holds. Given any precoloring $c : S^1 - W \rightarrow \{R, B\}$, Lemma 3.2 tells us that there are at least two non-forbidden extensions of c to $S^1 - \{w_0\}$. Let c^* be one such extension, which we may further extend to S^1 by choosing a color for w_0 . Assuming for the sake of contradiction that c cannot be extended to distinguish $O(2)$, Lemma 3.2 tells us that the two colorings c_R (obtained from coloring w_0 red) and c_B (obtained from coloring w_0 blue) are preserved by reflections τ_R, τ_B which do not fix w_0 . The rest of the proof can be taken almost word for word from [2], with the following caveats:

- 1) In the proof of Lemma 5 in [2], we know that $w_0 + \frac{1}{2} \notin W$ by $R(W)$.
- 2) In the proof of Lemma 12 in [2], the fact that $|\mathcal{O}_0| \geq 6$ (which depends on $T(W)$) is irrelevant to the proof of the lemma, and therefore can be omitted.

Otherwise, all arguments carry over exactly as written. ■

2.3 A weakening of condition $T(W)$

In order to prove Theorem 1, we will introduce another translational condition $T'(W)$, which is strictly weaker than $T(W)$, and show that $T'(W)$ implies $P(W)$.

Condition $T'(W)$: $(W + \frac{i}{k}) \cap W = \emptyset$ for $k = 4, 5$ and $\gcd(i, k) = 1$.

Theorem 2.3.1. *If $W \subset S^1$ of cardinality 4 satisfies $T'(W)$, then it also satisfies $P(W)$.*

Corollary 2.3.2. *$\text{ext}_D(C_n) = 4$ for all $n \geq 6$ such that $4 \nmid n$, $5 \nmid n$.*

Most of the necessary work for Theorem 2.3.1 involves checking that $P(W)$ holds in a few specific cases, which occurs in subsequent lemmas. We will first present a short argument that proves Theorem 2.3.1 assuming those lemmas, and prove the lemmas afterwards.

Proof of Theorem 2.3.1. Suppose $W \subset S^1$ of size 4 satisfies $T'(W)$. By Lemma 2.3.3, there are four possibilities: either $R(W)$ holds (in which case $P(W)$ holds by Theorem 2.2.2), W is $O(2)$ -equivalent to $\{0, \frac{1}{2}, a, a + \frac{1}{2}\}$ ($a \neq \pm \frac{1}{4}$), or W falls into one of two sporadic cases. Lemmas 2.3.4, 2.3.8, and 2.3.10 show that in each of the latter three cases, $P(W)$ holds, so we are done. \blacksquare

Lemma 2.3.3. *Suppose $W \subset S^1$ satisfies condition $T'(W)$. Then, either W satisfies $R(W)$, $W \cong \{0, \frac{1}{2}, a, a + \frac{1}{2}\}$ for some $a \neq \pm \frac{1}{4} \in S^1$, $W \cong \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$, or $W \cong \{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}\}$.*

Proof. Let $W \subset S^1$ be such that $|W| = 4$, $T'(W)$ holds, and $R(W)$ does not hold. Without loss of generality (by application of an automorphism of S^1), we may assume that $0 \in W$. Since $R(W)$ does not hold, we know that $\tau_0(W - \{0\}) \cap W \neq \emptyset$, which means that one of the following two statements is true.

- (1) $\frac{1}{2} \in W$
- (2) $\exists a \notin \{0, \frac{1}{2}\}$ such that $\{a, -a\} \subset W$

Suppose that $\frac{1}{2} \in W$. In this case, we have $W = \{0, \frac{1}{2}, a, b\}$ for some a and b . Then, τ_a does the following:

$$\begin{aligned} 0 &\mapsto 2a, \\ \frac{1}{2} &\mapsto 2a + \frac{1}{2}, \\ a &\mapsto a, \\ b &\mapsto 2a - b. \end{aligned}$$

Since $R(W)$ does not hold, we know that $\tau_a(W - \{a\}) \cap W = \emptyset$. However, we know that τ_a cannot fix 0 or $\frac{1}{2}$. If τ_a fixes b , then we have $b = a + \frac{1}{2}$, as claimed. Otherwise, τ_a must swap two of $\{0, \frac{1}{2}, b\}$; furthermore, possibly shifting W by $\frac{1}{2}$, we may assume that $\tau_a(0) = \frac{1}{2}$ or $\tau_a(0) = b$.

If $\tau_a(0) = \frac{1}{2}$, then we have $a = \frac{1}{4}$ or $\frac{3}{4}$, contradicting $T'(W)$. If $\tau_a(0) = b$, then $b = 2a$, and we consider τ_b . Again, we know that $|\tau_b(W) \cap W| \geq 2$, and by the same arguments as for τ_a , we may conclude that τ_b must swap two of $\{0, \frac{1}{2}, a\}$. Furthermore, τ_b cannot send 0 to $\frac{1}{2}$. If $\tau_b(0) = a$, then $4a = 2b = a \rightarrow 3a = 0$, which is one of the exceptions covered by the claim. Finally, if $\tau_b(\frac{1}{2}) = a$, then $4a + \frac{1}{2} = a \rightarrow 3a = \frac{1}{2}$, which is the last exception covered by the claim. Therefore, if $\frac{1}{2} \in W$, W does fall into one of the listed exceptions.

On the other hand, suppose that statement (1) is false; by translational symmetry, we may now assume that $(W + \frac{1}{2}) \cap W = \emptyset$. Furthermore, we know that $W = \{0, a, -a, b\}$ for some $a, b \notin \{0, \frac{1}{2}\}$. By the assumption that $R(W)$ does not hold, we know that $\tau_a(W - \{a\}) \cap W \neq \emptyset$; since we also assumed that $a + \frac{1}{2} \notin W$, there remain three possibilities: $\tau_a(0) = -a$, $\tau_a(0) = b$, or $\tau_a(-a) = b$.

If $2a = \tau_a(0) = -a$, then $3a = 0$ and by symmetry, we may assume that $a = \frac{1}{3}$. Then, τ_b cannot send one cube root of unity to another unless b is some 6th root of unity, as included in the list of exceptions.

If $2a = \tau_a(0) = b$, then we consider $\tau_{-a}(W) = \{-2a, -3a, -a, -4a\}$. Since $|\tau_{-a}(W) \cap W| \geq 2$, either $3a, 4a, 5a$, or $6a$ is equal to 0. But $3a = 0 \rightarrow b = 2a = -a$, a contradiction, while the second two subcases are impossible by $T'(W)$ and the fact that $\frac{1}{2} \notin W$. The final subcase is one of the listed exceptions.

If $3a = \tau_a(-a) = b$, then we consider τ_{-a} ; if it does not fall into either of the first two categories, then we obtain the opposite result: $-3a = b$ as well. But then $2b = 0$, a contradiction of Case 2

$((W + \frac{1}{2}) \cap W = \emptyset)$. Thus, the listed exceptions are in fact the only exceptions, as desired. \blacksquare

Lemma 2.3.4. *If $W = \{0, a, \frac{1}{2}, a + \frac{1}{2}\}$ for some $a \neq \pm \frac{1}{4}$, then $P(W)$ holds.*

Proof. Since $a \neq \pm \frac{1}{4}$, the collection $\{0, a, \frac{1}{2}, a + \frac{1}{2}, -a, -a + \frac{1}{2}\}$ consists of six distinct points. Let c be any precoloring of $S^1 - W$ such that $c(-a) = c(-a + \frac{1}{2}) = R$, and let d be any precoloring such that $d(-a) = R$, $d(-a + \frac{1}{2}) = B$ (by negating colorings, proving Lemma 2.3.4 in these two cases suffices to prove the (B, R) and (B, B) cases as well).

Extend c (respectively, d) to c_1 and c_2 (respectively, d_1 and d_2) in the following way: define $c_2(0) = d_1(0) = R$, $c_1(0) = d_2(0) = B$, $c_i(a) = d_i(a) = B$ for $i = 0, 1$, $c_i(\frac{1}{2}) = d_i(\frac{1}{2}) = R$, $c_1(a + \frac{1}{2}) = d_1(a + \frac{1}{2}) = B$, $c_2(a + \frac{1}{2}) = d_2(a + \frac{1}{2}) = R$.

Note 2.3.5. For k equal to c or d , k_1 and k_2 differ only on $W' := \{0, a + \frac{1}{2}\}$.

Note 2.3.6. $\tau_0, \tau_{\frac{1}{4}}, \tau_{\frac{a+\frac{1}{2}}{2}}, \sigma_{\frac{1}{2}}, \sigma_a$, and $\sigma_{a+\frac{1}{2}}$ do not preserve any of c_1, c_2, d_1, d_2 . Furthermore, $\tau_{\frac{a}{2}}$ does not preserve c_1, c_2 , or d_1 , and $\tau_{-\frac{a}{2}}$ does not preserve c_1, d_1 , or d_2 . In particular, if $\gamma \in O(2)$ preserves c_1, c_2, d_1 , or d_2 , then $\gamma(0) \notin W'$.

Finally, for $k = c$ or $k = d$, let the “intermediate coloring” k_3 be such that $k_3 = k_1 = k_2$ on $S^1 - \{0, a + \frac{1}{2}\}$, $k_3(0) = k_-(2a)$ (which is well-defined because $2a \notin \{0, a + \frac{1}{2}\}$), and $d_3(a + \frac{1}{2}) = d_3(0)$ while $c_3(a + \frac{1}{2}) = c_3(0)$.

We prove Lemma 2.3.4 by showing that one of $\{c_1, c_2, c_3\}$ is distinguishing, and one of $\{d_1, d_2, d_3\}$ is distinguishing. The arguments for c and d are extremely similar; we will have k denote either c or d and specify at which points the two arguments differ.

Assume that neither k_1 nor k_2 is distinguishing. Then, k_1 and k_2 must each be invariant under some nontrivial reflection or translation.

Suppose k_1 and k_2 are invariant under translations σ_1 and σ_2 . Then, we will use the fact that $\sigma_1\sigma_2(0) = \sigma_2\sigma_1(0)$ to derive a contradiction. By definition of σ_1 , $k_1(\sigma_1(0)) = k_1(0)$, which implies that $k_2(\sigma_1(0)) = k_1(0)$ unless $\sigma_1(0) \in W'$; this is not the case by Note 2.3.6.

Therefore, $k_2(\sigma_1(0)) = k_1(0)$. This then implies that $k_2(\sigma_2\sigma_1(0)) = k_1(0)$, which implies that $k_1(\sigma_2\sigma_1(0)) = k_1(0)$ except in the following two situations.

i) $\sigma_2\sigma_1(0) = 0$: This would mean that $\sigma_1(0) = \sigma_2^{-1}(0)$, which cannot happen because we know that $k_2(\sigma_1(0)) = k_1(0)$, but $k_2(\sigma_2^{-1}(0)) = k_2(0) \neq k_1(0)$.

ii) $\sigma_2\sigma_1(0) = a + \frac{1}{2}$: since $\sigma_1^2 \neq \sigma$ and $\sigma_1^2 \neq \text{id}$ (Note 2.3.6), we may suppose that $\sigma_2\sigma_1(0) \neq a + \frac{1}{2}$ by replacing σ_1 with σ_1^2 if necessary (it cannot be the case that $\sigma_2\sigma_1(0) = a + \frac{1}{2} = \sigma_2\sigma_1^2(0)$).

Since we can arrange that $\sigma_2\sigma_1(0) \neq a + \frac{1}{2}$, we can conclude that $k_1(\sigma_2\sigma_1(0)) = k_2(\sigma_2\sigma_1(0)) = k_1(0)$. Because this is entirely symmetric in σ_1, σ_2 , the same argument (except applying σ_2 first) proves that $k_1(\sigma_2\sigma_1(0)) = k_2(0)$, which is a contradiction.

Now suppose that k_1 is invariant under a translation σ and k_2 is invariant under a reflection τ . Again, by replacing σ with σ^2 if necessary, we can arrange that $\tau\sigma(0) \neq a + \frac{1}{2}$. We have the relation $\tau\sigma(0) = \sigma^{-1}\tau(0)$, which we will use to derive a contradiction. In fact, the previous argument fully carries over to allow us to conclude that $k_1(\tau\sigma(0)) = k_2(\tau\sigma(0)) = k_1(0)$, while $k_1(\sigma^{-1}\tau(0)) = k_2(\sigma^{-1}\tau(0)) = k_2(0)$. This argument also applies when k_1 is invariant under a reflection and k_2 is invariant under a translation (this is essentially a Red-Blue color swap).

Therefore, if neither k_1 nor k_2 is distinguishing, then they must be invariant under reflections τ_1 and τ_2 , respectively. For the remainder of the proof, let the translation $\sigma = \tau_2\tau_1$, and let k be defined on $S^1 - W'$ (i.e., extend it to a and $\frac{1}{2}$ because k_1 and k_2 match there). This step is inspired by the argument in [2] but takes it further – in [2], two reflections are composed in this way in a situation where their corresponding colorings differ at only one point. As a result, the orbits here are more complicated.

Observation 2.3.7. σ preserves k on $S^1 - \{0, \sigma^{-1}(0), \tau_1(0), a + \frac{1}{2}, \sigma^{-1}(a + \frac{1}{2}), \tau_1(a + \frac{1}{2})\}$. Additionally, k takes specific values as dictated by the following chart.

| | |
|---------------------------|----------|
| $\sigma^{-1}(0)$ | $k_2(0)$ |
| $\sigma(0)$ | $k_1(0)$ |
| $\tau_1(0)$ | $k_1(0)$ |
| $\tau_2(0)$ | $k_2(0)$ |
| $\tau_1(a + \frac{1}{2})$ | Blue |
| $\tau_2(a + \frac{1}{2})$ | Red |

Justifications. 1. For $\theta \in S^1$, if $\theta \notin W'$ and $\sigma(\theta) \notin W'$, then $k(\theta) = k(\sigma(\theta))$ (which is $k(\tau_2\tau_1(\theta))$) unless $\tau_1(\theta) \in W'$. The colors of the $\tau_i(0), \tau_i(a + \frac{1}{2})$ are dictated by the reflections' color-preserving properties.

2 (to show $k(\sigma(0)) = k_1(0)$). We know that $k(\sigma(0)) = k_1(0)$ unless $\tau_1(0) \in W'$ or $\sigma(0) \in W'$. But $\tau_1(0) \notin W'$ by Note 2.3.6, and $\sigma(0) \neq 0$ because if the opposite were true, then $\tau_1(0) = \tau_2(0)$, despite the fact that $\tau_1(0)$ and $\tau_2(0)$ must have opposite colors. Finally, if $\sigma(0) = a + \frac{1}{2}$, then $\sigma(\frac{1}{2}) = a$. We know that $k(a) = B$ and $k(\frac{1}{2}) = R$ for both $k = c$ and $k = d$. Therefore, there is a violation of σ color-preservation in either case (σ taking a red point to a blue point). We already have enough information to know that this happens at most at $\tau_1(0) \mapsto \tau_2(0)$ [for $k = c$, Red \mapsto Blue in fact never happens], so we would need that $\tau_1(0) = \frac{1}{2}$, which is impossible by Note 2.3.6.

3 (to show $k(\sigma^{-1}(0)) = k_2(0)$). The argument is similar here. As before, $\tau_2(0) \notin W'$ by Note 2.3.6, and we have already shown that $\sigma^{-1}(0) \neq 0$. If $\sigma^{-1}(0) = a + \frac{1}{2}$, then then we use the fact that then $\sigma(0) = -a + \frac{1}{2}$. We proved already that $k(\sigma(0)) = k_1(0)$, but in both the $k = c$ and $k = d$ cases, $k(-a + \frac{1}{2}) \neq k_1(0)$, a contradiction. Therefore, $k(\sigma^{-1}(0)) = k_2(0)$.

To complete the proof of Lemma 2.3.4, we now assume for the sake of contradiction that k_3 is also not distinguishing.

Again, we can see easily that k_3 cannot permit a nontrivial translation σ_3 , using the relation $\tau\sigma_3 = \sigma_3^{-1}\tau$ for $\tau \in \{\tau_1, \tau_2\}$. Pick $i \in \{1, 2\}$ such that $k_i(0) \neq k_3(0)$ (there is exactly one such i). On one hand, $\sigma_3(0) \notin W'$ (this can be easily checked), so $k(\sigma_3(0)) = k_3(\sigma_3(0)) = k_3(0)$ and therefore $k_i(\tau_i\sigma_3(0)) = k_3(0)$. On the other hand, we already know that $\tau_i(0) \notin W'$ (by Note 2.3.6) and so $k_3(\tau_i(0)) = k_i(\tau_i(0)) = k_i(0)$, and hence $k_3(\sigma_3^{-1}\tau_i(0)) = k_i(0)$. Since k_3 and k_i differ at only 0, this means that $\sigma_3\tau_i(0) = 0$, i.e., $\tau_i(0) = \sigma_3^{-1}(0)$; this contradicts the fact that $\tau_i(0)$ and $\sigma_3^{-1}(0)$ have opposite colors.

Therefore, we conclude that k_3 permits another reflection, τ_3 , and because k_3 permits neither τ_0 nor $\tau_{a+\frac{1}{2}}$ (we chose k_3 specifically so this is the case), we have that $\tau_3 \neq \tau_1, \tau_3 \neq \tau_2$. Thus, we have two nontrivial translations $\sigma_{31} := \tau_3\tau_1$ and $\sigma_{23} := \sigma_2\sigma_3$, satisfying the relation $\sigma_{23}\sigma_{31} = \sigma := \sigma_{21}$.

We will derive a contradiction using the fact that σ_{21} is also equal to $\sigma_{31}\sigma_{23}$ (i.e., the translations commute). Let $i \in \{1, 2\}$ be such that $k_i(0) \neq k_3(0)$, and let $j \in \{1, 2\}$ be such that $j \neq i$. Observation 2.3.7 tells us that $k(\sigma_{ji}(0)) = k_i(0)$. On the other hand, $\sigma_{ji}(0) = \sigma_{3i}\sigma_{j3}(0)$, and $k_j(0) = k_3(0)$. Since k_j and k_3 differ only at $a + \frac{1}{2}$, this means that $k_3(\sigma_{j3}(0)) = k_j(0)$ unless:

1) $\tau_3(0) = a + \frac{1}{2}$, i.e., $\tau_3 = \tau_{\frac{a+\frac{1}{2}}{2}}$. This does not hold, as we can easily note that $\tau_{\frac{a+\frac{1}{2}}{2}}(a) = \frac{1}{2}$ implies that this particular reflection does not preserve k_3 .

2) $\tau_j \tau_3(0) = a + \frac{1}{2}$. For $k = d$, this does not hold, because $\tau_3(0)$ has the same color as 0 under both k_j and k_3 , while $a + \frac{1}{2}$ has the opposite color under k_j . For $k = c$, this does not hold, because then we would have $\tau_j \tau_3(\frac{1}{2}) = a$. Since $k(\frac{1}{2}) = R$ and $k(a) = B$, this means that $\tau_3(\frac{1}{2}) = \tau_j(a) \in W'$. In particular, $c_j(\tau_j(a)) = B$ implies that $j = 1$, while $c_3(\tau_j(a)) = R$ implies that $\tau_3(\frac{1}{2}) = \tau_1(a) = a + \frac{1}{2}$. Then, we get a contradiction from the fact that $\tau_1 \tau_3(-a + \frac{1}{2}) = 0$, while $\tau_3(-a + \frac{1}{2}) = 2a + \frac{1}{2} \notin W'$ [$\tau_1 \tau_3$ sends the red $-a + \frac{1}{2}$ to the c_1 -blue 0].

Therefore, $k_3(\sigma_{j3}(0)) = k_j(\sigma_{j3}(0)) = k_j(0)$. Since $\sigma_{j3}(0) \neq 0$, this means that $k_i(\sigma_{j3}(0)) = k_j(0)$ as well. But this implies that $k_i(\tau_i \sigma_{j3}(0)) = k_j(0)$, and then we conclude that $k_i(\sigma_{3i} \sigma_{j3}(0)) = k_j(0)$ unless $\sigma_{3i} \sigma_{j3}(0) = 0$ (but $\sigma_{3i} \sigma_{j3} = \sigma_{ji}$, which we know is nontrivial) or $\tau_i \sigma_{j3}(0) = 0$ (which contradicts the fact that $k_i(\tau_i \sigma_{j3}(0)) = k_j(0) \neq k_i(0)$). Thus, $k_i(\sigma_{ji}(0)) = k_i(\sigma_{3i} \sigma_{j3}(0)) = k_j(0)$, contradicting Observation 2.3.7 (which says that $k(\sigma_{ji}(0)) = k_i(0)$). Hence, one of k_1, k_2 , and k_3 is distinguishing. This proves Lemma 2.3.4. \blacksquare

Lemma 2.3.8. *If $W = \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$, then $P(W)$ holds.*

Proof. The principles behind this proof are the same as those behind Lemma 2.3.4, and many of the same reductions are made. Let c be any precoloring of $S^1 - W$ such that $c(\frac{5}{6}) = c(\frac{1}{6}) = R$, and let d be any precoloring of $S^1 - W$ such that $d(\frac{5}{6}) = B$ and $d(\frac{1}{6}) = R$. Let $W' = \{0, \frac{2}{3}\}$, and extend c and d to c_1, d_1, c_2, d_2, c_3 , and d_3 in the following way:

| | c_1 | c_3 | c_2 | d_1 | d_3 | d_2 |
|---------------|-------|-------|-------|-------|-------|-------|
| 0 | B | R | R | B | B | R |
| $\frac{1}{3}$ | B | B | B | R | R | R |
| $\frac{1}{2}$ | B | B | B | B | B | B |
| $\frac{2}{3}$ | R | R | B | R | B | B |

In this proof, c_1, c_2 , and c_3 have the same purpose as they did in the proof of Lemma 2.3.4; that is, we first assume that none of c_1, c_2 , or c_3 is distinguishing, and show that c_1 and c_2 both permit translations τ_1, τ_2 . Then, we will show that the “intermediate” coloring c_3 also permits a translation τ_3 , and derive a contradiction from the (commutative) relation

$$\tau_2 \tau_1 = (\tau_2 \tau_3)(\tau_3 \tau_1) = (\tau_3 \tau_1)(\tau_2 \tau_3).$$

First, note the following things about the six colorings defined above.

Note 2.3.9. c_1 and d_1 are $\text{Aut}(C_6)$ -distinguishing. Considering $\{0, \pm\frac{1}{6}, \pm\frac{1}{3}, \frac{1}{2}\}$ as a copy of C_6 sitting inside S^1 ; the only element of $\text{Aut}(C_6)$ which fixes $c_2 : C_6 \rightarrow \{R, B\}$ is the reflection about 0; the only element of $\text{Aut}(C_6)$ which fixes d_2 is the reflection about $\frac{1}{6}$; the only element of $\text{Aut}(C_6)$ which fixes c_3 is the reflection about $-\frac{1}{12}$; the only element of $\text{Aut}(C_6)$ which fixes d_3 is the reflection about $\frac{1}{4}$. This means that no two of $\{c_1, c_2, c_3\}$ and $\{d_1, d_2, d_3\}$ can preserve the same reflection, because any such reflection would have to stabilize one element of W' , and no C_6 -reflection can preserve more than one of the listed colorings.

Let k be equal to c or d . Assume for the sake of contradiction that none of k_1, k_2 , or k_3 is $O(2)$ -distinguishing.

First, suppose that k_1 and k_2 are invariant under translations σ_1, σ_2 . We know that $k_1(\sigma_1(\frac{2}{3})) = R$ while $k_2(\sigma_2(\frac{2}{3})) = B$. This means that $k_2(\sigma_1(\frac{2}{3})) = R$ and $k_1(\sigma_2(\frac{1}{3})) = B$, because Note 2.3.9 tells us that $\sigma_i(W) \cap W = \emptyset$ (as any non-trivial translation which sends elements of C_6 to elements of C_6 does not even preserve color on $C_6 \subset S^1$). Applying σ_2 and σ_1 , respectively, we see that $k_2(\sigma_2 \sigma_1(\frac{2}{3})) = R$ while $k_1(\sigma_1 \sigma_2(\frac{2}{3})) = B$. Since $\sigma_1 \sigma_2 = \sigma_2 \sigma_1$, this is only possible if $\sigma_2 \sigma_1(\frac{2}{3}) \in W'$. On one hand, $\sigma_2 \sigma_1(\frac{2}{3}) \neq \frac{2}{3}$, because then $\sigma_1(\frac{2}{3}) = \sigma_2^{-1}(\frac{2}{3})$, contradicting the fact that $\sigma_1(\frac{2}{3})$ and $\sigma_2^{-1}(\frac{2}{3})$ (which are not elements of W') have different colors under k . On the other hand, $\sigma_2 \sigma_1(\frac{2}{3})$ cannot be equal to 0, because then $\sigma_2 \sigma_1(\frac{1}{6}) = \frac{1}{2}$; since $\sigma_1(\frac{1}{6}) \notin W$, this means that

$$R = k(\frac{1}{6}) = k(\sigma_1(\frac{1}{6})) = k(\sigma_2 \sigma_1(\frac{1}{6})) = k(\frac{1}{2}) = B,$$

a contradiction.

Next, suppose that k_1 is invariant under σ_1 and k_2 is invariant under a reflection, τ_2 . Then, we will use the fact that $\sigma_1\tau_2(\frac{2}{3}) = \tau_2\sigma_1^{-1}(\frac{2}{3})$ to derive a contradiction. Note 2.3.9 tells us that $\sigma_1(W) \cap W = \emptyset$, and either $\tau_2(0) \notin W'$ or $\tau_2(\frac{2}{3}) \notin W'$ (which one depends on whether $k = c$ or $k = d$). Therefore, we know that for some $w \in W'$,

$$k_2(w) = k_2(\tau_2(w)) = k_1(\tau_2(w)) = k_1(\sigma_1\tau_2(w))$$

while

$$k_1(w) = k_1(\sigma_1^{-1}(w)) = k_2(\sigma_1^{-1}(w)) = k_2(\tau_2\sigma_1^{-1}(w))$$

implying that $\sigma_1\tau_2(w) \in W'$ (because $k_1(w) \neq k_2(w)$). But $\sigma_1\tau_2(w) \neq w$, because then $\tau_2(w) = \sigma_1^{-1}(w)$, contradicting the fact that these points (which cannot be in W') have different colors under k .

Therefore, the only option is that $\sigma_1\tau_2(w)$ is equal to the other element of W' . But by replacing σ_1 with σ_1^2 (which is guaranteed to be different from σ and different from the identity), we can ensure that this does not happen, giving us our contradiction.

Since an argument analogous to the above will also work if k_1 were invariant under a reflection and k_2 were invariant under a translation, we conclude that k_1 is preserved by some reflection τ_1 and k_2 is preserved by τ_2 . Note 2.3.9 tells us that $\tau_1 \neq \tau_2$, and thus $\sigma_{21} := \tau_2\tau_1$ is a nontrivial translation.

Case 1. $k = d$. In this situation, we claim that $\sigma_{21}(\frac{2}{3}) \notin W'$ and $k(\sigma_{21}(\frac{2}{3})) = k_1(\frac{2}{3}) = R$.

To prove the claim, note that we know $R = k_1(\frac{2}{3}) = k_1(\tau_1(\frac{2}{3})) = k_2(\tau_1(\frac{2}{3})) = k_2(\tau_2\tau_1(\frac{2}{3}))$, because $\tau_1(\frac{2}{3}) \notin W'$ by Note 2.3.9. Therefore, the claim holds provided that $\sigma_{21}(\frac{2}{3}) \notin W'$. Since σ_{21} is nontrivial, we know that $\sigma_{21}(\frac{2}{3}) \neq \frac{2}{3}$. If $\sigma_{21}(\frac{2}{3}) = 0$, then we derive a contradiction from the fact that σ_{21} sends the $\frac{1}{2}, \frac{5}{6}, \frac{1}{6}$ triangle to itself. Note 2.3.9 tells us that σ_{21} would have to preserve k on this triangle (as no intermediate τ_1 -reflection could be an element of W'), but this triangle is not monochromatic under k . Thus, $\sigma_{21}(\frac{2}{3}) \notin W'$ and $k(\sigma_{21}(\frac{2}{3})) = R$, as desired.

Finally, we consider k_3 . If k_3 is preserved by some translation σ , then note that $\sigma_3^{-1}(\frac{2}{3}) \notin W'$ (no C_6 -translation preserves k_3), so

$$B = k_3(\frac{2}{3}) = k_3(\sigma_3^{-1}(\frac{2}{3})) = k_1(\sigma_3^{-1}(\frac{2}{3})) = k_1(\tau_1\sigma_3^{-1}(\frac{2}{3}))$$

which is equal to $k_2(\tau_1\sigma_3^{-1}(\frac{2}{3}))$ provided that $\tau_1\sigma_3^{-1}(\frac{2}{3}) \notin W'$. Since $\tau_1(\frac{2}{3})$ and $\sigma_3^{-1}(\frac{2}{3})$ have different colors under k , we know that $\tau_1\sigma_3^{-1}(\frac{2}{3}) \neq \frac{2}{3}$. By replacing σ_3 with σ_3^2 if necessary, we can ensure that $\tau_1\sigma_3^{-1}(\frac{2}{3})$ is not equal to 0, which allows us to conclude that $B = k_2(\tau_1\sigma_3^{-1}(\frac{2}{3})) = k_2(\sigma_{21}\sigma_3^{-1}(\frac{2}{3}))$. On the other hand, we know that $\sigma_{21}(\frac{2}{3}) \notin W'$ and $k(\sigma_{21}(\frac{2}{3})) = R$. This implies that $k_3(\sigma_3^{-1}\sigma_{21}(\frac{2}{3})) = R$, which is a contradiction (as $k_3(\frac{2}{3}) = k_3(0) = B$).

If k_3 is preserved by a reflection τ_3 , then Note 2.3.9 tells us that $\tau_3 \neq \tau_1$ and $\tau_3 \neq \tau_2$, meaning that $\sigma_{23} := \tau_2\tau_3$ and $\sigma_{31} := \tau_3\tau_1$ are nontrivial translations, and satisfy the relation

$$\sigma_{21} = \sigma_{23}\sigma_{31} = \sigma_{31}\sigma_{23}.$$

We already know that $\sigma_{21}(\frac{2}{3})$ is red, and not an element of W' . However, $\sigma_{23}(\frac{2}{3}) \neq \frac{2}{3}$ is certainly blue under k_2 because $\tau_3(\frac{2}{3}) \notin W'$ [this can be verified using Note 2.3.9]. Therefore, $\sigma_{23}(\frac{2}{3}) \neq 0$ (which is red under k_2), and hence σ_{31} sends a blue element of $S^1 - W'$ ($\sigma_{23}(\frac{2}{3})$) to a red element of $S^1 - W'$ ($\sigma_{21}(\frac{2}{3})$); this never happens (it sends the red $\tau_1(\frac{2}{3})$ to the blue $\tau_3(\frac{2}{3})$, but never the other way around). Hence, Case 1 leads to a contradiction.

Case 2. $k = c$. Here, we claim that $\sigma_{21}(0) \notin W'$ and $k(\sigma_{21}(0)) = B$. To see this, note that we know $B = k_1(0) = k_1(\tau_1(0)) = k_2(\tau_1(0)) = k_2(\tau_2\tau_1(0))$ [$\tau_1(0) \notin W'$ by Note 2.3.9], so the claim follows

if $\sigma_{21}(0) \notin W'$. We already know that $\sigma_{21}(0) \neq 0$, and if $\sigma_{21}(0) = \frac{2}{3}$, then σ_{21} sends the $\frac{1}{2}, \frac{5}{6}, \frac{1}{6}$ triangle to itself, giving us the same contradiction as in the $k = d$ case.

Now, just as in the $k = d$ case, k_3 cannot be preserved by a translation σ_3 , because then, after arranging for $\sigma_3\sigma_{21}(0) \notin W'$, we obtain a contradiction.

If k_3 is preserved by a reflection τ_3 , then we define σ_{32} and σ_{21} as before, and use the fact that, $\sigma_{21}(0)$, which is blue under all three extensions of k , is also equal to $\sigma_{31}\sigma_{23}(0)$. Furthermore, $\sigma_{23}(0)$ is red under k_2 , because $\tau_3(0) \notin W'$ [this can be checked using Note 2.3.9]. Therefore, $\sigma_{23}(0) \neq \frac{2}{3}$ (which is blue under k_2), and we conclude that σ_{31} sends a red element of $S^1 - W'$ to a blue element of $S^1 - W'$; this never happens; contradiction.

Thus, one of k_1, k_2 , and k_3 is distinguishing with respect to $O(2)$, which proves Lemma 2.3.8. ■

Lemma 2.3.10. *If $W = \{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}\}$, then $P(W)$ holds.*

Proof. This actually follows from the proof of Lemma 2.3.8 without any extra work. Instead of $W = \{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}\}$, we may equivalently consider $W = \{\frac{2}{3}, \frac{5}{6}, 0, \frac{1}{6}\}$. Let c be any precoloring of $S^1 - W$ such that $c(\frac{1}{3}) = c(\frac{1}{2}) = B$, and let d be any precoloring of $S^1 - W$ such that $c(\frac{1}{3}) = R$ and $c(\frac{1}{2}) = B$. Then, we may extend c and d to c_1, d_1, c_2, d_2, c_3 , and d_3 exactly as in the chart from Lemma 2.3.8. Since we already proved that one of $\{c_1, c_2, c_3\}$ and one of $\{d_1, d_2, d_3\}$ distinguishes $O(2)$, we are done. ■

This completes the proofs of the collection of lemmas necessary for Theorem 2.3.1.

2.4 Completing the proof of the Theorem 1

Having finished Theorem 2.3.1, we can loosen the translation constraint further and obtain an even better result for $|W| = 4$.

Condition $T''(W)$: $(W + \frac{i}{5}) \cap W = \emptyset$ for $\gcd(i, 5) = 1$.

Theorem 2.4.1. *If $W \subset S^1$, $|W| = 4$, and $T''(W)$ holds, then either $P(W)$ holds, or W is $O(2)$ -equivalent to $\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}\}$.*

Proof. Suppose $|W| = 4$, $T''(W)$ holds, and W is not equivalent to $\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}\}$. If $T'(W)$ holds, then by Theorem 2.3.1, $P(W)$ holds. If $T'(W)$ does not hold, then there are two possibilities.

Case 1. W is equivalent to $\{0, \frac{1}{4}, a, b\}$ for $a, b \notin \{\frac{2}{4}, \frac{3}{4}\}$. In this case, we claim that $R(W)$ holds. In particular, suppose that $R(W)$ does not hold. Then, since $\tau_0(\frac{1}{4}) = -\frac{1}{4} \notin W$ and $\frac{1}{2} \notin \{a, b\}$ (i.e., a and b are not fixed by τ_0), we must have that $b = -a$ (or else τ_0 would satisfy the desired property). But since $\tau_{\frac{1}{4}}(0) \notin W$ and $\frac{3}{4} \notin \{a, b\}$, we also must have that $b = -a + \frac{1}{2}$; a contradiction. Hence, $R(W)$ holds, and we conclude that $P(W)$ holds by Theorem 2.2.2.

Case 2. W is equivalent to $\{0, \frac{1}{4}, \frac{3}{4}, a\}$ for some $a \neq \frac{1}{2}$. Lemma 2.4.2 tells us that $P(W)$ holds in this situation (again, we will present the proof of the lemma below). Assuming this lemma, the proof of Theorem 2.4.1 is complete. ■

Lemma 2.4.2. *If $W \cong \{0, \frac{1}{4}, \frac{3}{4}, a\}$ for $a \neq \frac{1}{2}$, then $P(W)$ holds.*

Proof. The principles behind the argument are again similar to those in Lemma 2.3.4, Lemma 2.3.8, and Lemma 2.3.10. Let c be a precoloring of $S^1 - W$; by negating c if necessary, we may assume without loss of generality that $c(\frac{1}{2}) = B$. Then, extend c to c_1, c_2, c_3 in the following way: pick $c_1(a) = c_2(a) = c_3(a)$ such that exactly three of $\{a, -a, a + \frac{1}{2}, -a + \frac{1}{2}\}$ have the same color (this is always possible because either $-a, a + \frac{1}{2}$, and $-a + \frac{1}{2}$ all have the same color to begin with, or exactly two of the three have the same color). This choice guarantees that c_1, c_2 , and c_3 are not preserved by $\sigma_{\frac{1}{2}}$, τ_0 , or $\tau_{\frac{1}{4}}$. The rest of

W is colored according to this chart.

| | c_1 | c_3 | c_2 |
|---------------|-------|-------|-------|
| 0 | R | B | B |
| $\frac{1}{4}$ | B | B | R |
| $\frac{3}{4}$ | R | R | R |

Note that c_1 and c_2 only differ on $W' := \{0, \frac{1}{4}\}$. Also note that $\sigma_{\frac{1}{2}}, \sigma_{\frac{1}{4}}, \tau_0, \tau_{\frac{1}{4}},$ and $\tau_{\frac{3}{8}}$ do not preserve any of the c_i .. We will now run through the same argument as before – assuming that $c_1, c_2,$ and c_3 all satisfy some symmetry, showing that all of the symmetries must be reflections $\tau_1, \tau_2, \tau_3,$ and deriving a contradiction from the fact that $\tau_2\tau_1 = (\tau_2\tau_3)(\tau_3\tau_1) = (\tau_3\tau_1)(\tau_2\tau_3)$.

Assume for the sake of contradiction that $P(W)$ does not hold; in particular, we assume that none of $c_1, c_2,$ and c_3 are distinguishing. Suppose that c_1 and c_2 are both preserved by nontrivial translations σ_1, σ_2 . Then, $c_1(\sigma_1(0)) = c_1(0) = R$, which implies that $\sigma_1(0) \neq \frac{1}{4}$. Thus, $\sigma_1(0) \notin W'$, and hence $c_2(\sigma_2\sigma_1(0)) = c_2(\sigma_1(0)) = c_1(\sigma_1(0)) = R$. This argument is symmetric in c_1, c_2 , so we also conclude that $c_1(\sigma_2\sigma_1(0)) = B$. This can only happen if $\sigma_2\sigma_1(0) = \frac{1}{4}$, but this implies that $\sigma_2\sigma_1(\frac{1}{2}) = \frac{3}{4}$. Since $\sigma_1(\frac{1}{2}) \notin W'$ (as $\sigma_{\frac{1}{2}}$ and $\sigma_{\frac{1}{4}}$ do not preserve c_1), we know that $c(\sigma_1(\frac{1}{2})) = c_1(\frac{1}{2}) = B$. Then, we have σ_2 taking a blue element of $S^1 - W'$ to a red element of $S^1 - W'$; a contradiction. Therefore, it is not the case that c_1 and c_2 are both preserved by translations.

Now suppose that c_1 is preserved by a translation σ_1 and c_2 is preserved by a reflection τ_2 . Since $\tau_2(0) \notin W'$ (τ_0 and $\tau_{\frac{3}{8}}$ do not preserve c_1 or c_2), we know that $c(\tau_2(0)) = c_2(0) = B$, and hence $c_1(\sigma_1\tau_2(0)) = B$. Similarly, since $\sigma_1^{-1}(0) \notin W'$, we have that $c_2(\tau_2\sigma_1^{-1}(0)) = c(\sigma_1^{-1}(0)) = c_1(0) = R$. Since $\tau_2\sigma_1^{-1} = \sigma_1\tau_2$, this is only possible if $\tau_2\sigma_1^{-1}(0) = \frac{1}{4}$. But then we would have that $\tau_2\sigma_1^{-1}(\frac{1}{2}) = \frac{3}{4}$, which is a contradiction (lack of color preservation) because $\sigma_1^{-1}(\frac{1}{2}) \notin W'$. The same argument applies when c_1 is preserved by a reflection and c_2 is preserved by a translation.

We conclude that c_1 and c_2 are both preserved by reflections τ_1 and τ_2 . Since $\sigma_{\frac{1}{2}}$ and $\sigma_{\frac{1}{4}}$ also do not preserve c_3 , the argument from the previous paragraph also proves that c_3 must also be preserved by a reflection, τ_3 . Because none of the colorings are preserved by τ_0 or $\tau_{\frac{1}{4}}$, we know that $\tau_1, \tau_2,$ and τ_3 are pairwise distinct, allowing us to define nontrivial translations $\sigma_{ij} := \tau_i\tau_j$.

We already know that $\tau_i(W') \cap W' = \emptyset$ for all i , so we have that $c_i(\sigma_{ij}(0)) = c(\tau_j(0)) = c_j(0)$. Furthermore, we also know that $\sigma_{ij}(0) \notin W'$, because $\sigma_{ij}(0) = \frac{1}{4} \rightarrow \sigma_{ij}(\frac{1}{2}) = \frac{3}{4}$, which is only possible if $\tau_j(\frac{1}{2}) \in W'$. But $\tau_j(\frac{1}{2}) \neq 0$ (because $\tau_{\frac{1}{4}}$ never preserves a coloring) and $\tau_j(\frac{1}{2}) = \frac{1}{4} \rightarrow \tau_i(\frac{1}{4}) = \frac{3}{4}$, a contradiction (τ_0 never preserves a coloring). Thus, $c(\sigma_{ij}(0)) = c_j(0)$ for all $i \neq j$. In particular, $c(\sigma_{23}(0)) = B$ and $c(\sigma_{21}(0)) = R$. But $\sigma_{31}\sigma_{23}(0) = \sigma_{21}(0)$, so σ_{31} sends a blue element of $S^1 - W'$ to a red element of $S^1 - W'$. Since c_3 and c_1 differ only at 0 (where $c_1(0) = R$ and $c_3(0) = B$), it is easy to see that this is impossible. Thus, one of $c_1, c_2,$ and c_3 is distinguishing, which proves Lemma 2.4.2. ■

Corollary 2.4.3. *If $W \subset S^1$, $|W| = 5$, and $T''(W)$ holds, then $P(W)$ holds.*

Proof. If $W \subset S^1$, $|W| = 5$, and $T''(W)$ holds, then there is some subset W_4 of W of size 4 which is not equivalent to $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ (at worst, W can be equivalent to $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, a\}$, and we can remove 0, for instance). Then, by Theorem 2.4.1, $P(W_4)$ holds, and hence $P(W)$ holds. ■

Finally, we will remove all constraints on W to prove Theorem 1.

Theorem 2.4.4. *If $W \subset S^1$ and $|W| = 5$, then $P(W)$ holds unless $W \cong \{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$.*

Corollary 2.4.5. *If $W \subset S^1$ and $|W| = 6$, then $P(W)$ holds unconditionally.*

Proof. Let $W \subset S^1$ be such that $|W| = 5$ and W is not equivalent to the one exceptional set, and let W_4 be equal to *any* subset of W of size 4. Then, if $T''(W_4)$ holds, either $P(W_4)$ holds (in which

case $P(W)$ holds), or $W_4 \cong \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}\}$. In the latter case, let $a \in W - W_4$; then, $P(\{a\} \cup W_4 - \{\frac{2}{4}\})$ holds by Lemma 2.4.2. Therefore, $T''(W_4)$ cannot hold for any such W_4 . Then, there are two possibilities.

Case 1. $W \cong \{0, \frac{1}{5}, \pm \frac{2}{5}, a, b\}$ for some $a, b \in S^1$ [the \pm notation here means that *either* $\frac{2}{5} \in W$ or $-\frac{2}{5} \in W$, but not both]. In this case, without loss of generality, let a be such that $a \notin \{\pm \frac{3}{5}, \frac{4}{5}\}$ (either a or b must satisfy this property). Let $W_4 = \{0, \frac{1}{5}, \pm \frac{2}{5}, a\}$. We claim that $R(W_4)$ holds. To verify this, suppose that it were not the case. Since $\tau_0(\frac{1}{5}) = -\frac{1}{5}$ and $\tau_0(\pm \frac{2}{5}) = \mp \frac{2}{5}$ are not in W_4 , if $R(W_4)$ does not hold, then $-a \in W_4$. But we know that $-a \notin \{0, \frac{1}{5}, \pm \frac{2}{5}\}$, so we conclude that $-a = a$ and hence $a = \frac{1}{2}$. But then we can consider $\tau_{\frac{1}{5}}$ and $\tau_{\pm \frac{2}{5}}$; one of these reflections ($\tau_{\frac{2}{5}}$ in the $\frac{2}{5}$ case, and $\tau_{\frac{1}{5}}$ in the $-\frac{2}{5}$ case) satisfies $\tau(W_4 - \{a\}) \cap W_4 = \emptyset$. Therefore, if $R(W_4)$ does not hold, then we would need for $\tau(a) = a$ for this second reflection as well; since $\frac{1}{2}$ is not invariant under either $\tau_{\frac{1}{5}}$ or $\tau_{\frac{2}{5}}$, we conclude that $R(W_4)$ holds. As a result, Theorem 2.2.2 tells us that $P(W_4)$ holds, and thus $P(W)$ holds.

Case 2. $W \cong \{0, \frac{i_0}{5}, a, a + \frac{i_a}{5}, b\}$ for some $a, b \in S^1$ such that the $\sigma_{\frac{1}{5}}$ -orbits of $0, a$, and b are all distinct. In this case, let $W_3 = \{0, a, b\}$; if $\tau_0(W_3 - \{0\}) \cap W_3 = \emptyset$, then we define $W_4 = W_3 \cup \{\frac{i_0}{5}\}$. Since $\tau_{w_0}(\frac{i_0}{5}) = -\frac{i_{w_0}}{5}$ (which is not an element of W by the $\sigma_{\frac{1}{5}}$ -orbit condition), we conclude that $R(W_4)$ holds, and hence $P(W_4)$ holds by Theorem 2.2.2. The same reasoning applies if $\tau_a(W_3 - \{a\}) \cap W_3 = \emptyset$. If neither the τ_0 condition nor the τ_a condition holds, it is easily verified that $a = \frac{1}{2}$ or W_3 is equivalent to either $\{0, \frac{1}{4}, \frac{3}{4}\}$ or $\{0, \frac{1}{3}, \frac{2}{3}\}$. If $a = \frac{1}{2}$, then we let $W_4 = \{0, \frac{i_0}{5}, \frac{1}{2} + \frac{i_1}{5}, b\}$ and note that $R(W_4)$ holds (in particular, $\tau_0(W_4 - \{0\}) \cap W_4 = \emptyset$ by the $\sigma_{\frac{1}{5}}$ -condition). Hence, $P(W_4)$ holds by Theorem 2.2.2, and so $P(W)$ holds. If $W_3 \cong \{0, \frac{1}{4}, \frac{3}{4}\}$, then $P(W_3 \cup \{\frac{i_0}{5}\})$ holds by Lemma 2.4.2, and hence $P(W)$ holds. Finally, we can ensure that $W_3 \neq \{0, \frac{1}{3}, \frac{2}{3}\}$ by applying a translation by $\{-\frac{i_0}{5}\}$ if necessary (i.e., considering $\{\frac{i_0}{5}, a, b\}$ if $\{0, a, b\} = \{0, \frac{1}{3}, \frac{2}{3}\}$). Therefore, we may assume without loss of generality that $R(W_3)$ holds (in all cases in which we are not already done). Let $w_0 \in W_3$ be as described in condition $R(W_3)$, and let $W_4 = W_3 \cup \{w_0 + \frac{i_{w_0}}{5}\} \subset W$. Thus, since $R(W_4)$ holds, $P(W_4)$ holds by Theorem 2.2.2, and hence $P(W)$ holds in all cases. ■

2.5 An interesting corollary

As a result of our work above (in particular, from our proof of Lemma 2.3.4), we also get a result of a slightly different flavor.

Corollary 2.5.1. Let c be any 2-coloring of S^1 . Then, there exists a distinguishing coloring $c^* : S^1 \rightarrow \{R, B\}$ such that $c(x) \neq c^*(x)$ for at most three values of x .

Proof. Suppose we have a 2-coloring of S^1 , denoted c . If c is identically one color, then changing the colors of $0, \frac{1}{3}$, and $\frac{1}{2}$ suffices to produce a distinguishing 2-coloring.

If c is not identically one color, then we claim that there exists a reflection (about some point w_0) τ_{w_0} and a point $a \neq w_0 \pm \frac{1}{4}$ such that $c(\tau_{w_0}(a)) \neq c(a)$. If this were not the case, then, for example, $c(0)$ must be equal to $c(\theta)$ for all $\theta \neq \pm \frac{1}{4}$. Similarly, $c(\frac{1}{3})$ must be equal to $c(\theta)$ for all $\theta \neq \frac{1}{3} \pm \frac{1}{4}$. But then by transitivity we find that $c(0) = c(\theta)$ for all $\theta \in S^1$, contradicting the fact that c is not uniformly one color. This proves the claim.

Let τ_{w_0} be a reflection described in the claim. By translational symmetry, we may assume that $w_0 = 0$, so there exists some $a \neq \pm \frac{1}{4}$ such that $c(a) \neq c(-a)$. Let $W = \{0, a, \frac{1}{2}, a + \frac{1}{2}\}$, and let c' be the restriction of c to $S^1 - W$. Lemma 2.3.4 tells us that there exists an extension c^* of c' which is distinguishing; furthermore, the proof of Lemma 2.3.4 specifies that there exists a distinguishing c^* such that $c^*(a) \neq c'(-a) = c(-a)$. This means that $c^*(a) = c(a)$, so c^* and c differ at most on $\{0, \frac{1}{2}, a + \frac{1}{2}\}$. ■

3 Extending precolorings on \mathbf{R}^2 : a proof of Theorem 3

The complexity of extending precolorings on \mathbf{R}^2 is highly dependent on the choice of symmetry group Γ . First, we will show that the case of $\Gamma = O(2)$ has already been resolved by Theorem 1.

Theorem 3.1. $\text{ext}_D(\mathbf{R}^2, O(2)) = 7$.

Proof. The fact that 7 is a lower bound to the extension number was proved in [2]. Let $W \subset \mathbf{R}^2$ be such that $|W| = 7$ and the pointwise stabilizer $\text{stab}_{O(2)}(W)$ is trivial, and let c be a precoloring of $\mathbf{R}^2 - W$. Assume for the sake of contradiction that this precoloring cannot be extended to a distinguishing coloring of \mathbf{R}^2 . Note that the action of $O(2)$ on $\mathbf{R}^2 = \bigcup_{r \in \mathbf{R}_{\geq 0}} r \cdot S^1$ can be decomposed into separate actions of $O(2)$ on each individual $r \cdot S^1$. If there is any $r \in \mathbf{R}$ such that $|W \cap r \cdot S^1| \geq 6$, then $P(W, O(2))$ holds by Theorem 1. If not, then there are at least two nondegenerate circles in \mathbf{R}^2 which intersect W .

We now claim that there exist two points $x_1, x_2 \in W - \{0\}$ such that x_1 and x_2 are not on the same $r \cdot S^1$ and the line connecting x_1 to x_2 does not intersect $0 \in \mathbf{R}^2$. Suppose this were not the case. Let $C = r \cdot S^1$ be a nondegenerate circle such that $|C \cap W| > 0$ is minimal among circles that intersect W . If $C \cap W = \{x_1\}$, then if the claim is false, all of the other points in W lie on the line connecting 0 to x_1 ; this contradicts the stabilizer condition, because reflection across this line pointwise stabilizes W . On the other hand, if $C \cap W \supset \{x_1, x_2\}$ and the claim is false, we get that all elements of $W - C \cap W$ lie on the line connecting 0 to x_1 as well as the line connecting 0 to x_2 , showing that x_2 also lies on the line connecting 0 to x_1 . Applying this reasoning to all pairs of points inside $C \cap W$, we again obtain that W lies on a line, a contradiction. Thus, we may find x_1, x_2 as stated.

Let $x_1 \in r_1 \cdot S^1$ and $x_2 \in r_2 \cdot S^1$ be two points which satisfy the claim. Color the rest of W red (or any other combination of colors). Then, let c_1 be the coloring of $r_1 \cdot S^1$ where x_1 is red, and c_2 be the coloring where x_1 is blue. If c_1 satisfies an $SO(2)$ symmetry σ_1 and c_2 satisfies any $O(2)$ symmetry which does not fix x_1 , (σ_2 or τ_2) we obtain the usual contradiction from the relation $\sigma_1\sigma_2 = \sigma_2\sigma_1$ or $\sigma_1\tau_2 = \tau_2\sigma_1^{-1}$. The same holds where c_1 and c_2 are exchanged. Therefore, we may assume that c_1 satisfies a reflection τ_1 which fixes x_1 and no other symmetries, or both c_1 and c_2 satisfy reflections τ_1, τ_2 and no other symmetries.

In either case, c_1 satisfies only τ_1 , a reflection. If $\tau_1(x_2) \neq x_2$, we may color x_2 so that the final coloring c^* distinguishes $O(2)$, a contradiction. If $\tau_1(x_1) = x_1$, then by the claim, $\tau_1(x_2) \neq x_2$, so we are done. On the other hand, if $\tau_1(x_1) \neq x_1$, then c_2 satisfies only $\tau_2 \neq \tau_1$, so one of τ_1 or τ_2 must satisfy $\tau_i(x_2) \neq x_2$. Thus, we have proved Theorem 3.1. ■

The full isometry group $E(2) \supset O(2)$ is much more difficult to deal with. First, we'll classify the elements of $E(2)$, based on the identification $\mathbf{R}^2 \cong \mathbf{C}$: every $\gamma \in E(2)$ is either a translation ($z \in \mathbf{C} \mapsto z + a$), a rotation about some point ($z \mapsto \omega z + a$, $\omega \in S^1$, $a \in \mathbf{C}$), a reflection over some line ($z \mapsto \frac{-a}{\bar{a}}\bar{z} + a$, $a \in \mathbf{C}$), or a "glide reflection", which is a reflection over a line combined with a translation parallel to that line. The subgroup of $E(2)$ composed of translations and rotations is called $SE(2)$.

So far, we have used the work done on S^1 to prove $\text{ext}_D(\mathbf{R}^2, O(2)) = 7$. We can also apply the work from [2] done on $(\mathbf{R}, E(1))$ by considering the following subgroup Γ of $E(2)$: the group of translations $z \mapsto z + a$ and 180° rotations $z \mapsto -z + 2a$. By using the techniques from [2], we can prove the following lemma.

Lemma 3.2. Let $\Gamma \subset E(2)$ be as described above. Then, $\text{ext}_D(\mathbf{R}^2, \Gamma) = 4$.

Proof. The fact that 4 is a lower bound to the extension number follows from the fact that $\text{ext}_D(\mathbf{R}, E(1)) = 4$; whenever W is contained in a line ℓ , a precoloring of $\mathbf{R}^2 - W$ which colors $\mathbf{R}^2 - \ell$ red must be extended to distinguish the action of $E(1)$ on that line.

Let $W \subset \mathbf{R}^2$ be of size 4, and let c be a precoloring of $\mathbf{R}^2 - W$. Assume that c cannot be extended to a

distinguishing coloring of Γ . Recall the definition of the property $R(W)$ as it pertains to Γ : $R(W)$ holds if there exists some $w_0 \in W$ such that τ_{w_0} (which is now the 180° rotation about w_0) sends $W - \{w_0\}$ outside of W .

Claim. $R(W)$ always holds.

Proof. Among all $w \in W$ with minimal x -coordinate, pick the unique w_0 with minimal y -coordinate. Drawing axes perpendicular to the x and y -axes which meet at w_0 , it is clear that all of W sits inside the union of the right half-plane and the positive y -axis as defined by those axes. Therefore, $\tau_{w_0}(W)$ sits inside the left half-plane and the negative y -axis; this means that $\tau_{w_0}(W) \cap W \subset \{w_0\}$, proving the claim.

The rest of the proof of Lemma 3.2 follows exactly as the proof of Theorem 2.2.2, with no modifications. ■

We will now use Lemma 3.2 to prove the second half of Theorem 3. Let $SE(2) = \{\vec{v} \mapsto A\vec{v} + \vec{b}, A \in SO(2), \vec{b} \in \mathbf{R}^2\}$.

Theorem 3.3. $\text{ext}_D(\mathbf{R}^2, SE(2)) < \infty$.

Before proving the theorem, we first analyze the case of $|W| = 4$ in detail.

Lemma 3.4. Let $W \subset \mathbf{R}^2$ be such that $|W| = 4$, and let c be a precoloring of $\mathbf{R}^2 - W$. Suppose that c cannot be extended to a $SE(2)$ -distinguishing coloring of \mathbf{R}^2 . Then, there exists an extension c_0 of c which is Γ -distinguishing, and satisfies a 120° rotational symmetry.

Proof. Let W and c be as described in the statement of the lemma. The following technical lemma will be used to derive a contradiction.

Lemma 3.5. There exists an extension c_0 of c which is Γ -distinguishing, and preserved by some rotation γ_1 of either odd or infinite order (without loss of generality, about the origin $0 \in \mathbf{R}^2$). Furthermore, for at least one such c_0 , there exists an extension c_1 of c which is preserved under γ_1 and a point $x_0 \in W$ which is not fixed by γ_1 , such that the extension c_2 obtained from switching $c_1(x_0)$ to the opposite color is not preserved by either a rotation about x_0 or the map $z \mapsto -z + x_0$.

Proof: First, note that if we choose c_1 so that it is Γ -distinguishing and rotationally symmetric about 0, then changing $c_1(x_0)$ to the opposite color for any $x_0 \in W - \{0\}$ will never result in a rotational symmetry about x_0 . This is because if such a symmetry did result, then c_1 would itself be symmetric under some rotation about x_0 as well as another rotation about 0. The commutator of these two rotations is a nontrivial translation, so c_1 would not be E_1 -distinguishing. Therefore, we only have to deal with x_0 -rotational symmetry if c_1 is not chosen to be Γ -distinguishing.

Let c_0 be an extension of c to \mathbf{R}^2 which is Γ -distinguishing (Lemma 3.2 guarantees that c_0 exists). Since c_0 cannot be $SE(2)$ -distinguishing, c_0 must be preserved by some rotation which is not 180° . Without loss of generality, we may assume that the rotation is about the point $0 \in \mathbf{R}^2$, so c_0 is preserved by $\gamma_0 : z \mapsto \omega z$. Furthermore, γ_0 cannot have even order, for otherwise some power of γ_0 is the map $z \mapsto -z$, which we know does not preserve c_0 .

Now, suppose that for every $x \in W - \{0\}$, the coloring c_x obtained from changing $c_0(x)$ to the opposite color is preserved by $z \mapsto -z + x$.

Case 1. $0 \notin W$. Then for every $x \in W$, we have that $c_x(x) = c_x(0) = c_0(0)$, and so $c_0(x) \neq c_0(0)$. In other words, all of the points in W have the same color under c_0 . Now, let $x_1, x_2, x_3 \in W$ be such that $x_1 \neq 2^{\pm 1}x_2$ and $x_1 \neq 2^{\pm 1}x_3$ (three such points in W exist). Let c_1 be the coloring which matches c_0 except at x_1 and x_2 . Furthermore, by switching x_2 and x_3 if necessary (in the definition of c_1), we may assume that the 180° rotation about 0 does not preserve c_1 .

We want to show that c_1 satisfies the properties demanded by Lemma 3.5. In particular, we claim that c_1 is Γ -distinguishing.

To show this, suppose that c_1 is preserved by a translation $\sigma_1 : z \mapsto z + a$. By replacing a with a sufficiently large multiple of a , we can ensure that $a + x_1 - x_2 \notin \{x_1, x_2\}$ and $x_2 - a \neq x_1$ (so it makes sense to talk about $c(a + x_1 - x_2)$ and $c(x_2 - a)$). Furthermore, we know already that $x_1 - x_2 \notin \{x_1, x_2\}$ by our choice of x_1 and x_2 . Therefore, we see that $c(a + x_1 - x_2) = c(\sigma_1(x_1 - x_2)) = c(x_1 - x_2) = c_{x_1}(x_2) = c_0(x_2)$, but also that $c(a + x_1 - x_2) = c(-(x_2 - a) + x_1) = c(x_2 - a) = c_1(x_2) \neq c_0(x_2)$, a contradiction.

On the other hand, if c_1 is preserved by a 180° rotation $\gamma_1 : z \mapsto -z + a$, by the construction of c_1 we know that $a \notin \{0, x_1, x_2\}$. Furthermore, we know that $x_1 - x_2 \notin \{x_1, x_2\}$. Therefore, we see that $c_{x_1}(x_1 - a) = c(a) = c(0) = c_1(x_2) \neq c_{x_1}(x_2) = c(x_1 - x_2) = c_1(x_2 - x_1 + a)$. Now, $c_1(x_2 - x_1 + a) = c_{x_2}(x_2 - x_1 + a)$ as long as $x_2 - x_1 + a \neq x_1$, and we already know that $c_1(x_1) = c_1(x_2) \neq c_{x_1}(x_2)$, so this cannot happen. Therefore, we have that $c_{x_1}(x_1 - a) = c(0) \neq c_1(x_2 - x_1 + a) = c_{x_2}(x_2 - x_1 + a) = c_{x_2}(x_1 - a)$, which can only happen if $x_1 - a = x_1$, i.e., $a = 0$; a contradiction. This proves the claim that c_1 is Γ -distinguishing.

Since c_1 is Γ -distinguishing, we obtain γ_1 analogously to γ_0 as described before. Since two elements of W are colored red under c_1 and two elements are colored blue under c_1 , it is now certain that we can pick $x_0 \in W - \text{stab}_{\gamma_1}(\mathbf{R}^2)$ as the lemma describes.

Case 2. $0 \in W$. First of all, the argument from Case 1 still works almost all of the time. In this situation, we have that the three points in $W - \{0\}$ are all the same color (and opposite the color of 0) under c_0 . We can still flip the colors of $x_1, x_2 \in W - \{0\}$ to obtain another Γ -distinguishing coloring unless the following things both happen.

- 1) $W = \{0, \frac{1}{2}x, x, 2x\}$ for some x (meaning we can only find two points x_1, x_2 , and not a third).
- 2) $c_0(-\frac{1}{2}x) = c_0(-2x) = c_0(0)$ and $c_0(-x) = c_0(x)$ (when we change the colors of $\frac{1}{2}x_0$ and $2x_0$, a $z \mapsto -z$ symmetry results).

However, if (1) and (2) both happen, we note that $c_0(-\frac{1}{2}x) \neq c_0(x)$, contradicting the fact that flipping the color of $\frac{1}{2}x$ is supposed to result in a $z \mapsto -z + \frac{1}{2}x$ symmetry. Therefore, we can still find $x_1, x_2 \in W - \{0\}$ such that flipping the c_0 -colors of x_1 and x_2 results in an Γ -distinguishing coloring. Under this new coloring, which we call c^* , 0, x_1 , and x_2 all have the same color while x_3 (the last element of W) has the opposite color. As long as c^* is not rotationally symmetric about x_3 , we are again done.

Therefore, we may assume that c^* is symmetric under some rotation about x_3 . Then, pick c_1 to be the coloring which matches c_0 except at 0 (this is still symmetric under γ_0) and pick $x_0 = x_3$. Since c_1 itself is symmetric under $z \mapsto -z + x_3$, we know that c_2 (which is equal to c_1 except at x_3) is not symmetric under this rotation. Then, we are done unless c_2 is symmetric under some rotation about x_3 . This, combined with the fact that c^* is also symmetric under some rotation about x_3 , gives us a contradiction (from the usual commutation relation) unless 0, x_1 , and x_2 lie on a circle centered at x_3 such that 0, x_1 , and x_2 form an equilateral triangle.

Finally, this condition is actually symmetric under exchange of c_0 and c^* as the initial coloring (because both are Γ -distinguishing), so x_3 would also have to form an equilateral triangle with two out of $\{0, x_1, x_2\}$. However, the first triangle being equilateral and centered around x_3 rules out the possibility that any such second triangle could be equilateral; contradiction. This proves Lemma 3.5. \blacksquare

To finish the proof of Lemma 3.4, let c_1 , $\gamma_1 : z \mapsto \omega_1 z$, x_0 , and c_2 be as asserted by Lemma 3.5. Since we assumed that c is not $SE(2)$ -distinguishing, c_2 must be preserved by some $\gamma_2 : z \mapsto \omega_2 z + a$, where $\gamma_2(x_0) \neq x_0$ and $(\omega_2, a) \neq (-1, x_0)$ by the lemma. If $a = 0$, since c_1 and c_2 differ only at x_0 , we can derive the usual contradiction from the fact that $\gamma_1 \gamma_2 = \gamma_2 \gamma_1$; therefore, we may suppose that $a \neq 0$. Now, note that the commutator $\sigma_1 = \gamma_2^{-1} \gamma_1^{-1} \gamma_2 \gamma_1$ is a translation, $z \mapsto z + \frac{a(1-\omega_1)}{\omega_1 \omega_2}$. Since we

know that $a \neq 0$ and $\omega_1 \neq 1$, we in fact know that this is a nontrivial translation. Similarly, we have that $\sigma_2 := \gamma_1^{-2}\gamma_2^{-1}\gamma_1^2\gamma_2 : z \mapsto z - \frac{a(1-\omega_1^2)}{\omega_1^2\omega_2}$ is a nontrivial translation, as $a \neq 0$ and $\omega_1^2 \neq 1$. Therefore, we have the relation $\sigma_1\sigma_2 = \sigma_2\sigma_1$, which in terms of γ_1 and γ_2 becomes

$$\gamma_2^{-1}\gamma_1^{-1}\gamma_2\gamma_1^{-1}\gamma_2^{-1}\gamma_1^2\gamma_2 = \gamma_1^{-2}\gamma_2^{-1}\gamma_1\gamma_2\gamma_1.$$

Without loss of generality, suppose that $c_1(x_0) = R$. Noting that $\sigma_1\sigma_2(x_0) = x_0 + \frac{a(\omega_1-1)}{\omega_1^2\omega_2} \neq x_0$, we will show that, under the caveat that γ_1 may be replaced by γ_1^n for some n and γ_2 may be replaced with γ_2^2 , $c(\sigma_1\sigma_2(x_0)) = B$ while $c(\sigma_2\sigma_1(x_0)) = R$.

We start by looking at the right side of the equation (and applying each letter of the word one at a time). We know that $\gamma_1(x_0) = \omega_1 x_0 \neq x_0$, and hence $c(\gamma_1(x_0)) = R$. Therefore, we have that $c(\sigma_2\sigma_1(x_0)) = R$ unless one of the following two things happens:

- 1) $\gamma_1\gamma_2\gamma_1(x_0) = x_0$.
- 2) $\sigma_2\sigma_1(x_0) = x_0$ (which we have already ruled out).

It does not matter if, say, $\gamma_2\gamma_1(x_0) = x_0$, because then $\gamma_2\gamma_1(x_0)$ will still be red under c_1 , and we can continue applying the next letter. Writing out an explicit formula for $\gamma_1\gamma_2\gamma_1$ then tells us that $c(\sigma_2\sigma_1(x_0)) = R$ unless $\omega_1^2\omega_2x_0 + a\omega_1 = x_0$, i.e., $x_0 = \frac{a\omega_1}{1-\omega_1^2\omega_2} := A_1(\gamma_1, \gamma_2)$.

Now, analyzing the left hand side of the equation, we know that $\gamma_2(x_0) \neq x_0$ by Lemma 3.5, so $c(\gamma_2(x_0)) = B$. Therefore, we have that $c(\sigma_1\sigma_2(x_0)) = B$ unless one of the following two things happens:

- 1) $\gamma_2^{-1}\gamma_1^2\gamma_2(x_0) = x_0$, which reduces to the equation $x_0 = \frac{-a}{\omega_2} := A_3(\gamma_1, \gamma_2)$.
- 2) $\gamma_2\gamma_1^{-1}\gamma_2^{-1}\gamma_1^2\gamma_2(x_0) = x_0$, which reduces to the equation $x_0 = \frac{a(\omega_1^2+\omega_1-1)}{\omega_1(1-\omega_1\omega_2)} := A_2(\gamma_1, \gamma_2)$.

Thus, we obtain our contradiction unless $x_0 = A_i(\gamma_1, \gamma_2)$ for some $1 \leq i \leq 3$. Now, we show that with the proper modifications, we can ensure that this does not happen.

Step 1. Possibly replacing γ_2 with γ_2^2 , we can ensure that $x_0 \neq A_3(\gamma_1, \gamma_2)$.

Suppose that $x_0 = \frac{-a}{\omega_2}$. Then, as $\gamma_2^2(z) = \omega_2^2 z + \omega_2 a + a$, we see that $A_3(\gamma_1, \gamma_2^2) = \frac{-a(\omega_2+1)}{\omega_2^2}$, and if $x_0 = A_3(\gamma_1, \gamma_2^2)$ as well, we obtain that either $a = 0$ or $\omega_2 = \omega_2 + 1$; a contradiction in either case. Therefore, as long as γ_2^2 does not fix x_0 , we can safely replace γ_2 with γ_2^2 . Furthermore, since γ_2^2 is a rotation about some $y \neq x_0$, γ_2^2 fixes x_0 if and only if either $\omega_2 = -1$ or γ_2^2 is the identity (which also implies that $\omega_2 = -1$). In this situation, since $x_0 = \frac{-a}{\omega_2}$, we get that $x_0 = a$, which (combined with $\omega_2 = -1$) cannot be the case by Lemma 3.5. Thus, we may assume without loss of generality that $x_0 \neq A_3(\gamma_1, \gamma_2)$. Furthermore, since $A_3(\gamma_1, \gamma_2)$ is actually independent of γ_1 , any future modifications to γ_1 will not change this fact.

Step 2. Attempt to replace γ_1 with γ_1^2 .

Since γ_1 does not have even order, γ_1^2 also does not have even order, so we may repeat the above process using γ_1^2 instead of γ_1 . Therefore, we obtain our desired contradiction unless $x_0 = A_i(\gamma_1, \gamma_2)$ for some $1 \leq i \leq 2$ and $x_0 = A_j(\gamma_1^2, \gamma_2)$ for some $1 \leq j \leq 2$. This gives us four pairs of simultaneous equations in the three variables ω_1, ω_2, a , whose solutions (simplified using $a \neq 0$ and $\omega_1 \notin \{0, 1\}$) are as follows:

$$\begin{aligned} B_{1,1}(\omega_1, \omega_2, a) &:= \omega_2 + \frac{1}{\omega_1^3} = 0, \\ B_{1,2}(\omega_1, \omega_2, a) &:= \omega_1^3 + \omega_1 + 1 = 0, \\ B_{2,1}(\omega_1, \omega_2, a) &:= \omega_2 + \frac{\omega_1^2 - 1}{\omega_1^4(\omega_1 + 2)} = 0, \end{aligned}$$

$$B_{2,2}(\omega_1, \omega_2, a) := \omega_2 + \frac{\omega_1^3 + 1}{\omega_1(\omega_1^2 - \omega_1 - 1)} = 0.$$

First, note that for a fixed ω_2 , there are at most $3 + 3 + 5 + 3 = 14$ values of ω_1 which satisfy any of the four above equations. Leaving ω_2 fixed, we may now replace ω_1 (respectively, γ_1) with ω_1^n (γ_1^n) for any $n \in \mathbf{Z}$ unless $\omega_1^{2n} = 1$ (as then γ_1^{2n} is the identity). If ω_1 has infinite order in S^1 , then there are infinitely many elements in $\{\omega_1^n, n \in \mathbf{Z} - \{0\}\}$, and so there exists some value of n for which $B_{i,j}(\omega_1^n, \omega_2, a) \neq 0$ for all $i, j \in \{1, 2\}$; this in turn implies that we get our contradiction.

Thus, ω_1 has order N for some $N \in \mathbf{Z}_{>0}$, and Lemma 3.5 tells us that N is odd. It can be easily checked (by computer, for example) that $B_{1,2}(\omega_1, \omega_2, a) = 0$ has no solutions over the N th roots of unity, so we are further restricted to only considering $B_{1,1}$, $B_{2,1}$, and $B_{2,2}$. Now, replacing ω_1 with ω_1^{-1} must still leave one of $B_{1,1}$, $B_{2,1}$ or $B_{2,2}$ satisfied (else we are done); this gives us nine pairs of simultaneous equations which all have explicit solutions for $\omega_1 \in \mathbf{C}$. Checking by computer, we find that the only solutions for ω_1 within the odd roots of unity are when ω_1 is a cube root of unity. Lemma 3.5 tells us that there exists an extension c_0 of c which is Γ -distinguishing and preserved by γ_1 , so we have now proved Lemma 3.4. \blacksquare

Lemma 3.4 gives us very specific information about when precolorings of $\mathbf{R}^2 - \{\text{four points}\}$ cannot be extended to distinguish $SE(2)$. We can now use this to prove Theorem 3.3.

Proof of Theorem 3.3. Let $N \gg 0$, let $|W| \subset \mathbf{R}^2$ be such that $|W| = N$, and let c be a precoloring of $\mathbf{R}^2 - W$. Assume for the sake of contradiction that c cannot be extended to distinguish $SE(2)$. Then, Lemma 3.4 tells us that there exists an extension c_0 of c which is Γ -distinguishing, and invariant under some order 3 rotation γ_0 . Identify a set $Y = \{y_1, y_2, y_3, y_4\}$ of four special points such that Y contains no equilateral triangle. Let $W' = W - Y$. Now, from W' pick a set of four points $S_4 = \{x_4^{(i)}, 1 \leq i \leq 4\}$, and three additional points x_1, x_2, x_3 . There are $F(N) = \binom{N-4}{4}(N-8)(N-9)(N-10)$ such ordered collections (S_4, x_1, x_2, x_3) . On the other hand, there are only $O(\frac{F(N)}{N})$ such ordered collections such that $S_4 \cup \{x_1, x_2, x_3\} \cup Y$ contains any equilateral triangle (because for any pair of points (x_1, x_2) , there are only two points in \mathbf{R}^2 which form an equilateral triangle with them).

For any collection (S_4, x_1, x_2, x_3) such that $S_4 \cup \{x_1, x_2, x_3\} \cup Y$ does not contain any equilateral triangle, for each $1 \leq j \leq 3$, let c_j^* be the precoloring of $\mathbf{R}^2 - \{y_1, y_2, y_3, y_4\}$ which differs from c_0 at exactly x_j , and let c_4^* be the precoloring which differs from c_0 on exactly S_4 . By Lemma 3.4, there exists an extension c_j of c_j^* which is Γ invariant and symmetric under some rotation γ_j of order 3; we can arrange that the γ_j are all of the form $z \mapsto e^{\frac{2\pi i}{3}} z + a_j$. Let $(S_4, x_1^{(1)}, x_2, x_3)$ and $(S_4, x_1^{(2)}, x_2, x_3)$ be two such collections. If $\gamma_1^{(1)} = \gamma_1^{(2)}$, then $\gamma_1^{(1)}$ fixes two colorings $c_1^{(1)}$ and $c_1^{(2)}$ which differ for at least two points $(x_1^{(1)}$ and $x_1^{(2)})$; thus, since γ_1 has order 3, $Y \cup \{x_1^{(1)}, x_1^{(2)}\}$ must contain an equilateral triangle. Since $Y \cup \{x_1^{(1)}\}$ does not contain an equilateral triangle, for any fixed $x_1^{(1)}$, there are at most k possible values of $x_1^{(2)}$ for which $\gamma_1^{(1)} = \gamma_1^{(2)}$ could be the case, where k is small and independent of N . The same holds true for varying x_2 and x_3 .

We know that γ_4 has exactly one fixed point; therefore, at least three elements of S_4 are not fixed by γ_4 . Furthermore, for each of these three $x_4^{(j)}$, either $\gamma_4(x_4^{(j)}) \notin S_4 \cup Y \cup \{x_3\}$ or $\gamma_4^{-1}(x_4^{(j)}) \notin S_4 \cup Y \cup \{x_3\}$ (this is because $S_4 \cup Y \cup \{x_3\}$ contains no equilateral triangle). Without loss of generality, this means that $\gamma_4^{-1}(x_4^{(j)}) \notin S_4 \cup Y \cup \{x_3\}$ for $1 \leq j \leq 2$ (two of the three have to satisfy the property for the same power of γ_4 , which can be γ_4^{-1} without loss of generality). Then, among these two $x_4^{(j)}$, pick x_4 such that $\gamma_4^{-1}(x_4) \neq \gamma_3^{-1}(x_3)$. Thus far, we have simply picked one element of S_4 , given a fixed collection (S_4, x_1, x_2, x_3) .

Again because $Y \cup S_4 \cup \{x_1, x_2, x_3\}$ contains no equilateral triangle, we may suppose that $\gamma_2^{-1}(x_4) \notin Y \cup S_3 \cup \{x_1, x_2, x_3\}$ (the power of γ_2 does not matter). We will now find a collection (S_4, x_1, x_2, x_3) which gives us a contradiction from the relation $\gamma_1 \gamma_2^{-1} \gamma_3 \gamma_4^{-1} = \gamma_3 \gamma_4^{-1} \gamma_1 \gamma_2^{-1}$ (by construction, $\gamma_1 \gamma_2^{-1}$ and $\gamma_3 \gamma_4^{-1}$ are translations).

First, note that we have already shown that $\gamma_4^{-1}(x_4) \notin S_4 \cup Y \cup \{x_3\}$, so $c_3(\gamma_3\gamma_4^{-1}(x_4)) = c_3(\gamma_4^{-1}(x_4)) = c_4(\gamma_4^{-1}(x_4)) = c_4(x_4)$. Now $c_3(\gamma_3\gamma_4^{-1}(x_4)) = c_2(\gamma_3\gamma_4^{-1}(x_4))$ as long as $\gamma_3\gamma_4^{-1}(x_4) \notin Y \cup \{x_2, x_3\}$. We already know that $\gamma_3\gamma_4^{-1}(x_4) \neq x_3$. Let S_4, x_1 , and x_2 be fixed. Then, the equations $\gamma_3\gamma_4(x_4) = x^*$ for $x^* \in Y \cup \{x_2\}$ each have exactly one solution for $\gamma_3 : z \mapsto \omega z + a_3$ (i.e., we can solve the linear equation in a_3), so there are at most five such “bad” rotations. We also showed that there are at most k choices for x_3 yielding any given rotational symmetry, so there are at most $5k$ choices of x_3 that are potentially problematic. In total, this means that at most $5k \binom{N-4}{4} (N-8)(N-9) = O(\frac{F(N)}{N})$ ordered collections which are problematic at this step.

Suppose that our collection is not one of those problematic collections. Then, our work so far has shown that $c_2(\gamma_2^{-1}\gamma_3\gamma_4^{-1}(x_4)) = c_2(\gamma_3\gamma_4^{-1}(x_4)) = c_4(x_4)$. We may change the c_2 on the left hand side of this equation to a c_1 as long as $\gamma_2^{-1}\gamma_3\gamma_4^{-1}(x_4) \notin Y \cup \{x_1, x_2\}$. For any $x^* \in Y \cup \{x_1, x_2\}$, the equation $\gamma_3\gamma_4^{-1}(x_4) = \gamma_2(x^*)$ yields at most $6k$ more “bad” choices of x_3 given a fixed (S_4, x_1, x_2) , and again at most $O(\frac{F(N)}{N})$ problematic collections.

Supposing that our collection is still not problematic, our work so far shows that $c_1(\gamma_1\gamma_2^{-1}\gamma_3\gamma_4^{-1}(x_4)) = c_4(x_4)$. To ensure that $\gamma_1\gamma_2^{-1}\gamma_3\gamma_4^{-1}(x_4) \notin Y \cup \{x_1, x_3\}$, by the same argument as in the previous paragraphs, we have to remove from consideration $O(\frac{F(N)}{N})$ problematic collections – the only difference here is that we impose a constraint on the choice of x_2 given any fixed S_4, x_1, x_3 . Thus, for all but $O(\frac{F(N)}{N})$ of the $F(N)$ possible collections, we find that $c_3(\gamma_1\gamma_2^{-1}\gamma_3\gamma_4^{-1}(x_4)) = c_4(x_4)$.

We can use the same exact argument to show that for all but $O(\frac{F(N)}{N})$ of the $F(N)$ collections, $c_3(\gamma_3\gamma_4^{-1}\gamma_1\gamma_2^{-1}(x_4)) = c_2(x_4)$; the only point at which there could be a problem is if $\gamma_2^{-1}(x_4) = x_2$, but we have already arranged for this not to be the case. Therefore, we obtain a contradiction by using any of $F(N) - O(\frac{F(N)}{N})$ collections (S_4, x_1, x_2, x_3) ; thus, for sufficiently large N , all sets W such that $|W| \geq N$ satisfy $P(W, SE(2))$. This completes the proof of Theorem 3.3. ■

4 Infinite extension numbers in higher dimensions

In [2], it was conjectured that $\text{ext}_D(\mathbf{R}^2, E(2)) = 7$, $\text{ext}_D(S^2, O(3)) = 9$, and $\text{ext}_D(\mathbf{R}^3, E(3)) = 10$. The authors of [2] also posed the question of computing these extension numbers in higher dimensions. In Section 8, we focused on the \mathbf{R}^2 case, where some progress was made; however, the conjecture from [2] remains open. On the other hand, once we go beyond \mathbf{R}^2 to $X = \mathbf{R}^n$ ($n \geq 3$) or $X = S^n$ ($n \geq 2$), we now show that the extension number $\text{ext}_D(X, \text{Isom}(X))$ is always infinite (indeed, we will see in Section 5.2 that this is even the case for small subgroups of $\text{Isom}(X)$). However, we separate the case of S^2 from the others, because for $X = \mathbf{R}^n$ and $X = S^n$ when $n \geq 3$, we can give very explicit uncountable sets W and precolorings of $X - W$ which cannot be extended.

4.1 A proof of Theorem 2

Example 4.1.1. Let $X = \mathbf{R}^n$ for $n \geq 4$, and let $\{e_1, \dots, e_n\}$ be the standard basis for X . Let $W = \mathbf{R}e_1 \cup \{e_2, e_3, \dots, e_n\}$, and note that the pointwise stabilizer of W inside $E(n)$ is trivial, because any isometry which fixes the standard basis of \mathbf{R}^n fixes all of \mathbf{R}^n .

Claim. $P(W, E(n))$ does not hold.

Proof. Let c be the precoloring of $X - W$ which is uniformly red, and let c^* be any extension of c to X . Then, since $|\{e_2, \dots, e_n\}| \geq 3$, there exist e_i and e_j with $1 \notin \{i, j\}$ such that $c^*(e_i) = c^*(e_j)$. Furthermore, there exists $\gamma \in O(n) \subset E(n)$ such that $\gamma(e_i) = e_j$, $\gamma(e_j) = e_i$, and γ pointwise stabilizes the orthogonal complement of $\mathbf{R}e_i + \mathbf{R}e_j$. Since e_i and e_j are the only possible elements of $\mathbf{R}e_i + \mathbf{R}e_j$ not colored red, we immediately obtain that γ fixes c^* . ■

Example 4.1.2. Let $X = \mathbf{R}^3$, and let $\{e_1, e_2, e_3\}$ be the standard basis. Considering $\mathbf{R}e_1 + \mathbf{R}e_2 \cong \mathbf{R}^2$, let T be the vertices of an equilateral triangle in $\mathbf{R}e_1 + \mathbf{R}e_2$ centered at the origin. Then, let $W = T \cup \mathbf{R}e_3$. Since W linearly spans all of \mathbf{R}^3 , it has trivial pointwise stabilizer.

Claim. $P(W, O(3))$ does not hold.

Proof. Let c be the precoloring of $\mathbf{R}^3 - W$ which is uniformly red, and let c^* be any extension of c to \mathbf{R}^3 . Let S^1 be the copy of the unit circle inside \mathbf{R}^3 which contains T . Since $\text{Isom}(S^1)$ does not distinguish this coloring of S^1 , there exists some planar reflection or rotation which fixes $c^*|_{S^1}$. Let $\gamma \in O(3)$ act by this reflection or rotation on $\mathbf{R}e_1 + \mathbf{R}e_2$ and stabilize e_3 (such an element of $O(3)$ certainly exists). Then, because γ stabilizes $\mathbf{R}e_3$ and preserves $c^*|_{S^1}$, since $c^*(\mathbf{R}^3 - \mathbf{R}e_3 \cup S^1) = R$, we conclude that γ preserves c^* . ■

Example 4.1.3. Let $X = S^n \subset \mathbf{R}^{n+1}$ with $n \geq 3$, and let e_1, \dots, e_{n+1} be the standard basis of \mathbf{R}^{n+1} . Let S^1 be the copy of the unit circle sitting inside S^n with coordinates $x_3 = x_4 = \dots = x_{n+1} = 0$, and let $T \subset S^1$ be the vertices of an equilateral triangle. Finally, let S^{n-2} be the orthogonal complement of S^1 sitting inside S^n , and let $W = S^{n-2} \cup T$. Since W linearly spans \mathbf{R}^{n+1} , it has trivial pointwise stabilizer within $O(n+1)$.

Claim. $P(W, O(n+1))$ does not hold.

Proof. This is essentially the same as Example 4.1.2. Letting c be the precoloring of $S^n - W$ which is uniformly red, any extension c^* of c to S^n , when restricted to S^1 , is preserved by some planar rotation or reflection. Then, we may let $\gamma \in O(n)$ be the isometry which is equal to this rotation or reflection when restricted to S^1 and pointwise stabilizes S^{n-2} . By construction, γ preserves c^* . ■

4.2 Extending precolorings on S^2 : a proof of Theorem 4

Since all of the counterexamples from Section 5.1 involved invariance under some reflectional symmetry, it is reasonable to ask if removing reflections from the isometry groups would give us finite extension numbers. While one can create counterexamples in \mathbf{R}^6 very similar to Example 4.1.1 which do not satisfy $P(W, SO(6))$ (and similarly in higher dimensions), we can also employ another method to create huge numbers of counterexamples in lower dimensions – even on S^2 , where Section 5.1 failed to produce any results.

Let us recall the statement of Theorem 4.

Theorem 4. *If $W \subset S^2$ is finite, then $P(W, SO(3))$ does not hold. Assuming the axiom of choice, we may replace “finite” with “countable.”*

In particular, if we let $SO(3)$ act on \mathbf{R}^n by acting on a particular copy of $\mathbf{R}^3 \subset \mathbf{R}^n$, Theorem 4 implies that $\text{ext}_D(\mathbf{R}^n, SO(3)) = \infty$. In some sense, this is a much stronger result than Theorem 2, because it produces a huge class of counterexamples. On the other hand, it does not produce any uncountable sets W such that $P(W)$ does not hold.

Proof of Theorem 4. Let $W \subset S^2$ be any finite set, with $|W| = n$. We will construct a precoloring of $S^2 - W$ which cannot be extended to distinguish $SO(3)$. First, we’ll establish a framework to make the problem easier to think about.

Let $\Gamma \subset SO(3)$ be a subgroup with generating set S , by which we mean that the elements of S and the elements of S^{-1} together generate Γ as a group. Then, we can produce a graph $G(W, \Gamma, S)$ as follows: let $V(G) = \Gamma \cdot W$ and $E(G) = \{(x, y, s) \in V(G) \times V(G) \times S : y = s^{\pm 1}x \text{ for some } s \in S\}$. In other words, each edge is labelled by some element of S , and there may be more than one edge connecting two vertices. We say that a 2-coloring c of $G(W, \Gamma, S)$ is *invariant* under a particular $s \in S$ if all s -adjacent vertices (that is, pairs (x, y) such that $(x, y, s) \in E(G)$) have the same color under c .

We will construct a “bad” precoloring c of $S^2 - W$ by picking a group Γ (with generating set S) such that the graph $G(W, \Gamma, S)$ has a particularly nice structure. In particular, we will use the result, attributed to Hausdorff in [4] (1914), that $SO(3)$ contains a copy of F_2 , the free group on two letters.

There are explicit constructions of free subgroups of $SO(3)$; for example, in [6], it is shown that rotations about the angle ϕ in the x - y plane and in the x - z plane generate a free group provided that $\cos(\phi) \in \mathbf{Q} - \{0, \pm 1, \pm \frac{1}{2}\}$. Furthermore, it is well known that F_2 contains as a subgroup $F_{\mathbf{N}}$, the free group on countably infinite many letters (for example, see [5]). Let $\Gamma' = F_{\mathbf{N}} \subset SO(3)$.

Claim. There exists a free subgroup $\Gamma \subset \Gamma'$ with infinite generating set S such that the following two statements are true: (1) the connected components of the elements $w_i \in W$ inside $G(W, \Gamma, S)$ are pairwise disjoint (i.e., there are no paths between elements of W), and (2) $G(W, \Gamma, S)$ contains no cycles which contain any element of W .

Proof. We first find a subgroup satisfying property (2). Let $S' = \{s_1, s_2, \dots\}$ be the free generating set of Γ' . Let $w_1 \in W$. Then, $\text{stab}_{\Gamma'}(w_1)$ is a subgroup of Γ' which is also abelian, because $\text{stab}_{SO(3)}(\{w_1\})$ is abelian. We know that abelian subgroups of a free group are isomorphic to \mathbf{Z} ; this is a special case of the Nielsen-Schreier theorem (which relies on the axiom of choice), but this special case does not rely on the axiom of choice (for example, see [5]). Therefore, $\text{stab}_{\Gamma'}(\{w_1\})$ is generated by a single $\gamma \in \Gamma'$ that can be written as a word in finitely many letters $s_{i_1}, s_{i_2}, \dots, s_{i_k}$. Letting $S'_1 = S' - \{s_{i_1}, s_{i_2}, \dots, s_{i_k}\}$ and $\Gamma'_1 = \langle S'_1 \rangle$, we see that $\text{stab}_{\Gamma'_1}(\{w_1\})$ is trivial. Repeating this process for each of the elements of W , we obtain the subgroup $\Gamma'_n \subset \Gamma'$ with infinite generating set S'_n such that $\text{stab}_{\Gamma'_n}(\{w\})$ is trivial for any $w \in W$ - in other words, (Γ'_n, S'_n) satisfies property (2).

To prove the full claim, for any pair $w_i, w_j \in W$, we note that there is at most one element $\gamma \in \Gamma'_n$ such that $\gamma w_i = w_j$; this is because if there were two such elements, γ_1, γ_2 , then $\gamma_1 \gamma_2^{-1} \in \text{stab}_{\Gamma'_n}(w_j)$, which is trivial. Since there are finitely many elements of W , there are finitely many γ 's in total, each of which is a word in finitely many generators $s_{j_1}, s_{j_2}, \dots, s_{j_l}$. Letting $S = S'_n - \{s_{j_1}, \dots, s_{j_l}\}$ and $\Gamma = \langle S \rangle$, we see that (Γ, S) satisfies both (1) and (2), as desired in the claim. ■

Let (Γ, S) be as asserted in the claim. Since S is infinite, let s_1, \dots, s_{2^n} be 2^n elements of s . We will construct a precoloring c of $S^2 - W$ such that, enumerating the extensions of c by c_1, \dots, c_{2^n} , c_i is fixed by s_i for each $1 \leq i \leq 2^n$. We will construct c as follows: color $\mathbf{R}^2 - G(W, \Gamma, S)$ red. Thus, it will suffice to show in the end that each of the extensions, when restricted to $G(W, \Gamma, S)$, is invariant under some s_i . By property (1), we may consider (and color) the connected components of each $w \in W$ separately.

Enumerate the colorings of W by $c_1^*, c_2^*, \dots, c_{2^n}^*$, and let $w \in W$. First, we consider the vertices $x \in V(G(W, \Gamma, S))$ which are adjacent to w - in other words, $x = s_i^{\pm 1} w$ for some $i \in \mathbf{N}$. If $i > 2^n$, then define $c(x) = R$. If $i \leq 2^n$, define $c(x) = c_i^*(w)$. Now, since $G(W, \Gamma, S)$ contains no cycles that contain w , for every $y \in \Gamma\{w\}$, there exists a unique “branch” $x \in V(G(W, \Gamma, S))$ which is adjacent to w such that every path from y to w has x as its second to last vertex. Then, define $c(y) = c(x)$. Under this construction of c , it is clear that $c_i(x) = c_i(w)$ if x is s_i -adjacent to w . Furthermore, if y_1 and y_2 are s_i -adjacent, then it is clear that y_1 and y_2 have the same branch x , so $c(y_1) = c(x) = c(y_2)$. Thus, the coloring c_i is invariant under s_i for each $1 \leq i \leq 2^n$, as desired.

For W finite, we managed the above construction without invoking the axiom of choice. For W countably infinite, we can do the same exact construction, but we must use the following fact proved by de Groot and Dekker in [3]: assuming the axiom of choice, $SO(3)$ contains a free group F on uncountably many letters.

We need to use this fact because if W is countably infinite, we may need to remove countably many generators from the generating set of F to remove all of the cycles that contain elements of W , and more importantly, there are uncountably many colorings of W which need to satisfy some symmetry. However, replacing “finite” with “countable” and “countable” with “uncountable” as necessary, the above

argument will construct a precoloring of $S^2 - W$ which cannot be extended to distinguish F , or $SO(3)$. ■

Finally, we note that Theorem 4 remains true even if we consider k -colorings for $k > 2$. In [2], the distinguishing extension number is defined for any $k \geq D_\Gamma(X)$ (rather than just $k = D_\Gamma(X)$), but $\text{ext}_D(S^2, O(3), k) = \infty$ for every $k \geq 2$.

5 Open questions

Of the conjectures and questions posed in [2], the conjecture that $\text{ext}_D(\mathbf{R}^2, E(2)) = 7$ remains open. We pose a weakened version of this conjecture, as well as another related conjecture.

Conjecture 5.1. $\text{ext}_D(\mathbf{R}^2, E(2)) < \infty$.

Conjecture 5.2. $\text{ext}_D(\mathbf{R}^2, SE(2)) = 4$.

We can also ask if Theorem 2 (which holds for S^n and \mathbf{R}^n for $n \geq 3$) applies to S^2 .

Question 5.3. Do there exist uncountable subsets $W \subset S^2$ such that $P(W, O(3))$ does not hold? Such that $P(W, SO(3))$ does not hold? At least in the case of $SO(3)$, we are inclined to believe that this is not the case.

Finally, the motivation for introducing the distinguishing extension number was to better differentiate group actions on sets – the distinguishing number $D_\Gamma(X)$ is very often 1, 2, or 3, for example. However, $\text{ext}_D(X, \Gamma)$ cannot differentiate between $O(3)$ acting on \mathbf{R}^3 and $O(4)$ acting on \mathbf{R}^4 , among other things. One possible alternative to the extension number is the following.

Definition. The *replacement number* $R(X, \Gamma)$ is the smallest $n \in \mathbf{N}$ such that for every $D_\Gamma(X)$ -coloring of X , we may replace the colors of at most n points in X to obtain a distinguishing coloring of X .

For example, Corollary 17.7 states that $R(S^1, O(2)) = 3$. It is easy to further establish that $R(\mathbf{R}, E(1)) = 3$ as well. To make sure that $R(X, \Gamma)$ is not bounded in terms of the distinguishing number (in an obvious way, at least), we note that $R(\mathbf{R}^n, E(n)) \geq n$, because any $n - 1$ points lie on some hyperplane (so the all-red coloring cannot be fixed using $n - 1$ points). Since the “replacement” constraint is considerably weaker than the “extension” constraint, we are led to the following conjecture.

Conjecture 5.4. $R(\mathbf{R}^n, E(n)) < \infty$.

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